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ABSTRACT

Unit 14 in the SMSG secondary school mathematics series is a student text covering the following topics in geometry: areas of polygonal regions, similarity, circles and spheres, characterization of sets, constructions, areas of circles and sectors, volumes of solids, and plane coordinate geometry. Appendices cover Eratosthenes' measurement of the earth, rigid motions, proof of the two-circle theorem, trigonometry, and regular polyhedra. (DT)

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STUDENT'S TEXT

UNIT NO. 14

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SCHOOL MATHEMATICS STUDY GROUP

YALE UNIVERSITY PRESS

School Mathematics Study Group

Geometry

Unit 14

Geometry

Student's Text, Part II

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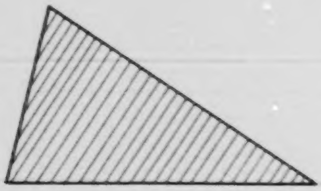
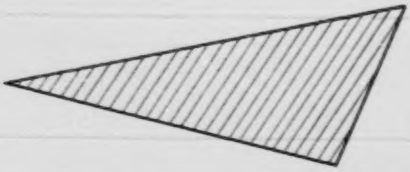
Chapter 11

AREAS OF POLYGONAL REGIONS

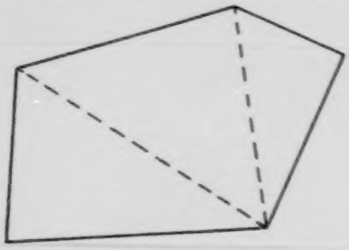
11-1. Polygonal Regions.

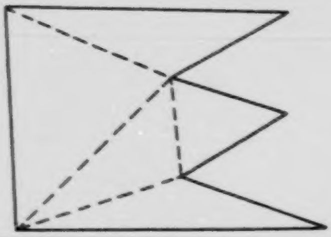
A triangular region is a figure that consists of a triangle plus its interior, like this:





A polygonal region is a figure in a plane, like one of these:

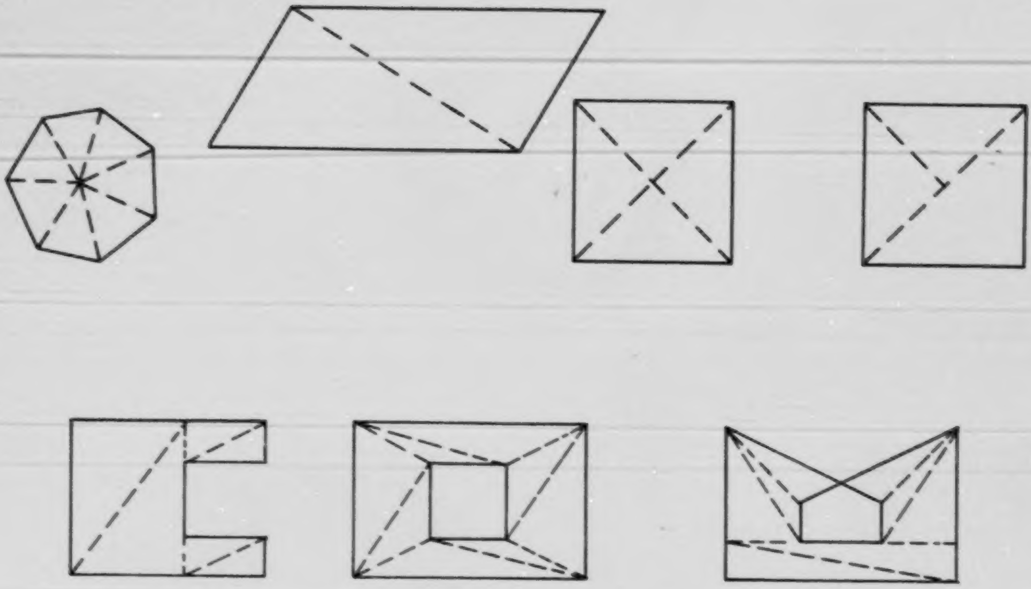




that can be "cut up" into triangular regions. To be exact:

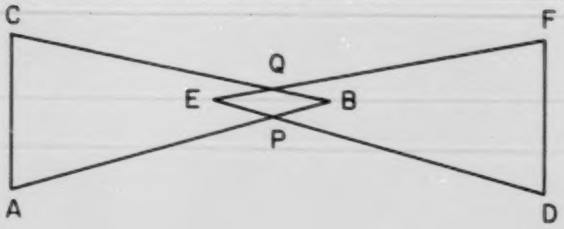
Definitions: A triangular region is the union of a triangle and its interior. A polygonal region is the union of a finite number of coplanar triangular regions, such that if any two of these intersect the intersection is either a segment or a point.

The dotted lines in the figures above show how each of the two figures can be cut up in this way. Here are more examples:

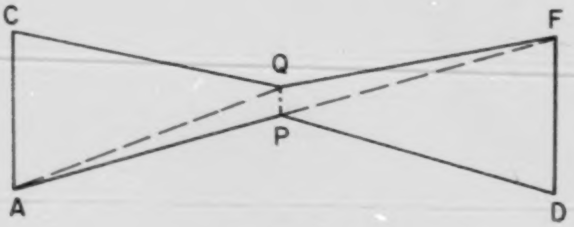


In the last two examples the figures have "holes" in them. This possibility is not excluded by the definition, and these figures are perfectly good polygonal regions.

On the other hand, the region APDFQC cannot be "cut up" into regions ABC and DEF even though it *is* the union of these two triangular regions. The intersection of the two triangular regions is the quadrilateral region EPBQ, which is certainly not a segment or a point. This does not mean that APDFQC is not a polygonal region, but merely that its description as a union of triangular regions ABC and DEF is not enough to

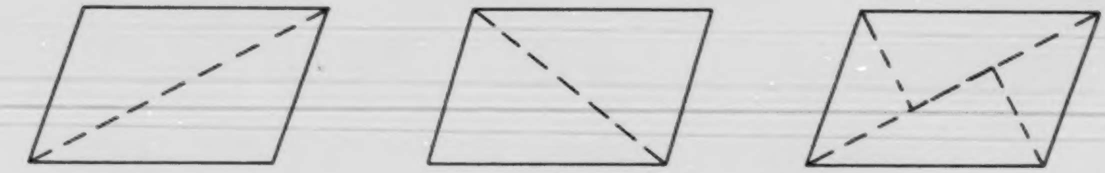


show this. APDFQC is in fact a polygonal region, as is shown below.



The polygonal regions form a rather large class of figures. Of course, there are simple and important figures that are not polygonal regions. For example, the figure formed by a circle together with its interior is not of this type.

If a figure can be cut up into triangular regions, then this can be done in a great many ways. For example, a parallelogram plus its interior can be cut up in many ways. Here are three of these ways.



In this chapter we will study the areas of polygonal regions, and learn how to compute them. The sixteen postulates that we have introduced so far would enable us to do this, but the treatment would be extremely difficult and quite unsuitable for a beginning geometry course like this one. Instead we shall introduce measure of area in much the same way we did for measure of distance and angle, by means of appropriate postulates.

Postulate 17. To every polygonal region there corresponds a unique positive number.

Definition: The *area* of a polygonal region is the number assigned to it by Postulate 17.

We designate the area of a region R simply by *area* R. In the following postulates, when we speak of a region, for short, it would be understood that we mean a polygonal region.

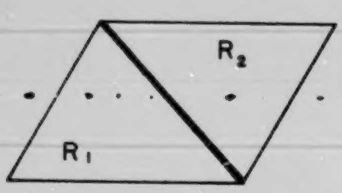
Our intuition tells us that two regions of the same shape and size should have the same area, regardless of their positions in space. This fundamental fact is the motivation of the next postulate.

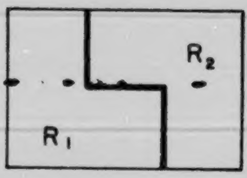
Postulate 18. If two triangles are congruent, then the triangular regions have the same area.

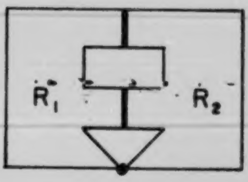
If a region is cut into two pieces it is clear that the area of the region should be the sum of the areas of the pieces. This is what our next postulate says. Let us state the postulate and then consider its meaning.

Postulate 19. Suppose that the region R is the union of two regions R_1 and R_2. Suppose that R_1 and R_2 intersect at most in a finite number of segments and points. Then the area of R is the sum of the areas of R_1 and R_2.

The three figures below show examples of the application of this Postulate.

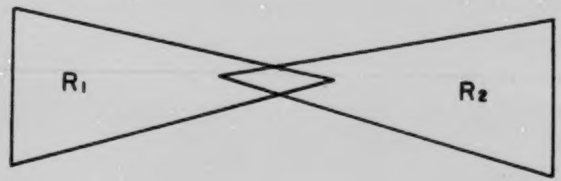




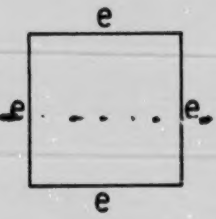


In each figure the intersection is heavily marked, and consists of a segment in the first figure, three segments in the second, and two segments and a point in the third.

On the other hand, the next figure is the union of two triangular regions, R_1 and R_2, but their intersection is not made up of a finite number of segments and points. Instead it is the quadrilateral region in the middle. Thus Postulate 19 cannot be applied to this case. If we tried to calculate the area of the whole region by adding the areas of R_1 and R_2 the area of the quadrilateral region would be counted twice. It was in anticipation of this situation that we insisted, in the definition of polygonal region, that the triangles determining the region must not overlap.



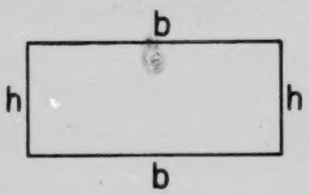
As was the case with distance and angle, the "unit of area" can be specified at will. However, it is convenient and customary to choose this unit to be closely associated with the unit of distance. If we are to measure distance in inches, we measure area in square inches; and in general, whatever unit of distance we use, we use the corresponding square unit to measure area. One way to ensure this would be to state as a postulate that the area of a square is to be the square of the length of an edge.



(By the "area of a square" we mean, of course, the area of the polygonal region which is the union of the square and its interior. We will speak in the same way of the area of any quadrilateral, meaning the area of the corresponding polygonal region.)

The statement $A = e^2$ is, however, a little too special to be convenient. The difficulty is that if we establish our unit of area by the postulate $A = e^2$, then we would have the problem of proving that the corresponding formula holds also for rectangles. That is, we would have to prove that the area of a rectangle is the product of the length of its base and the length of its altitude. Of course, if we know that this holds for rectangles, then it follows immediately that for squares we have $A = e^2$, because every square is a rectangle. The converse can also be proved, but the proof is harder than one might think. The most convenient thing to do, for the present, is to take as a postulate the more general formula, that is, the one for rectangles:

> Postulate 20. The area of a rectangle is the product of the length of its base and the length of its altitude.



$A = bh$

✓

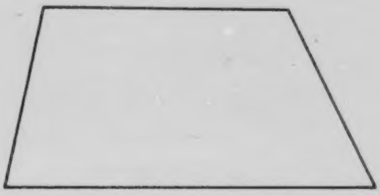
Notice that in the previous paragraph and in Postulate 20 we were very careful to say, "length of its base" and "length of its altitude". In using Postulate 20 from now on, we will just say, "The area of a rectangle is the product of its base and its altitude". This means that we use "base" and "altitude" sometimes to indicate line segments and sometimes to indicate their lengths. From now on we will do this fairly generally, trusting in your ability to tell from the context which meaning of a word we intend. If we "bisect a side of a triangle" the word "side" will have its original meaning, as a set of points. If we "square the side of a triangle" we are using the word "side" as an abbreviation for "length of the side". Such abbreviations will be very convenient in this and later chapters.

On the basis of the four area postulates we can calculate the areas of triangles, parallelograms, and a variety of other figures.

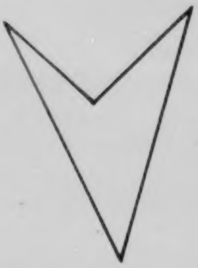
Problem Set 11-1

1. Show that each of the regions below is polygonal by indicating how each can be cut into triangular regions such that if two of them intersect their intersection is a point or segment of each of them. Try to find the smallest number of triangular regions in each case.

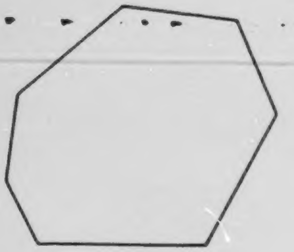
a.



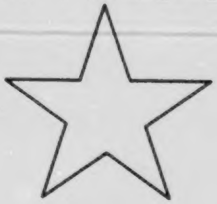
b.



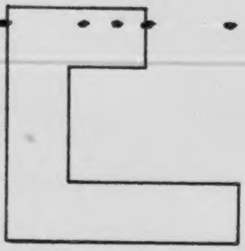
c.



d.



e.



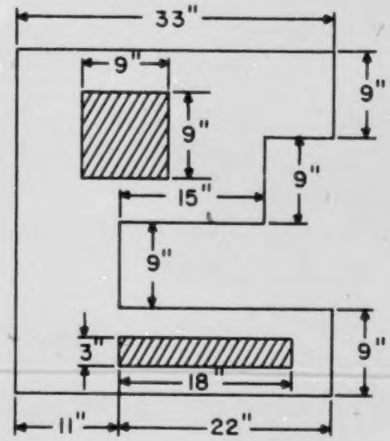
2. Find the area of a rectangle 50 ft. long and $16\frac{1}{2}$ ft. wide.

3. a. If you double the altitude of a rectangle and leave the base the same, how is the area changed?

   b. If both the altitude and the base of a rectangle are doubled, how is the area changed?

4. How many tiles, each 6 inches square, does it take to cover a rectangular floor 37 ft. 6 in. by 12 ft.?

5. The figure shown is a face of a certain machine part. In order to compute the cost of painting a great number of these parts it is necessary to know the area of a face. The shaded regions are not to be painted. Find the area to be painted.



*6. Are the following statements true or false? Give a reason for each answer.

a. A triangle is a polygonal region.

b. Postulate 17 says that for every positive number A there corresponds some polygonal region R.

c. Every polygonal region has an unique area.

d. If two triangles are congruent, then the triangular regions have the same area.

e. The union of two polygonal regions has an area equal to the sum of the areas of each region.

f. Postulate 20 assures us that the area of a square having side e is $A = e^2$.

g. The interior of a trapezoid is a polygonal region.

h. A triangular region is a polygonal region.

7. A rectangular region having base 6 and altitude 4 can be divided up into squares having a base 2, as in Figure 1. Notice that a square with base 2 is the largest square possible which will divide the rectangular region into an exact number of congruent squares.

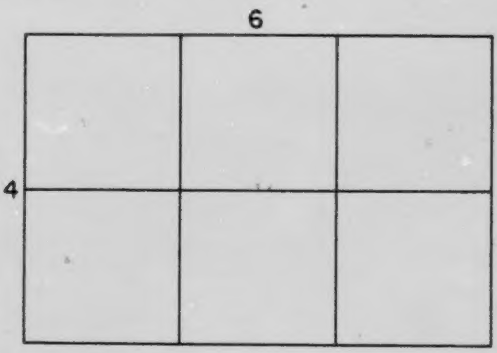


Figure 1.

Similarly, a square with base $\frac{1}{2}$ is the largest square possible which will exactly divide a rectangular region with base 4 and altitude $1\frac{1}{2}$, as in Figure 2.

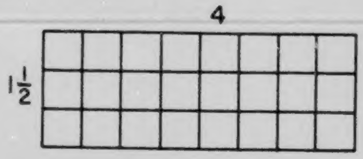


Figure 2.

Determine the side of the largest square which will exactly divide rectangular regions having the following measures:

a. $b = 4$; $h = 12$.

b. $b = 5$; $h = 2\frac{3}{4}$.

c. $b = 3.5$; $h = 1.7$.

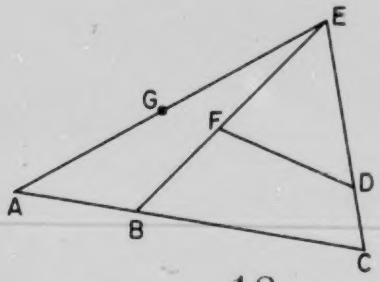
d. $b = 1.7$; $h = 1.414$.

e. $b = 2.0$; $h = \sqrt{2}$.

f. $b = \sqrt{2}$; $h = \sqrt{3}$.

What difficulty do you find in parts (e) and (f)? Do you see that this relates to the discussion of the text preceding Postulate 20?

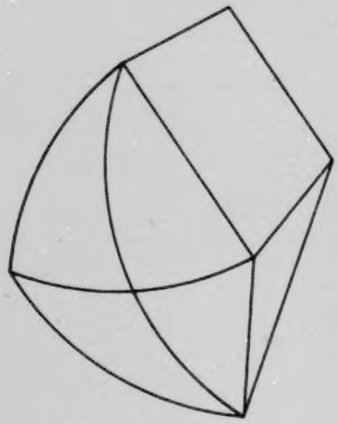
*8. In the following figure, A, B, C, D, E, F, G are called vertices, the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EG}$, $\overline{GA}$, $\overline{EF}$, $\overline{FD}$, $\overline{FB}$ are called edges, and the polygonal regions ABE, FED, BCDF are called faces. The exterior of the figure will also be considered as a face.



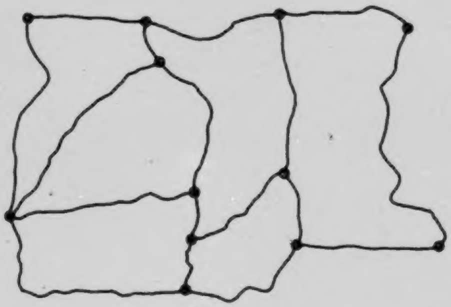
Let the number of faces be f, the number of vertices be v, and the number of edges be e. In a theorem originated by a famous mathematician, Euler, the following formula occurs: f - e + v, which refers to figures of which the above figure is one possibility. Using the figure, let's compute f - e + v. You should see that f = 4, v = 7, e = 9, and this gives us f - e + v = 2.

Using the two figures below, compute f - e + v. Notice that the edges are not necessarily segments.

a.



b. Suppose this figure to be a section of a map showing counties:



c. What pattern do you observe in the results of the three computations?

d. In part (a) take a point in the interior of the quadrilateral and draw segments from each of the four vertices to the point. How does this affect the computation of f - e + v? Can you explain why?

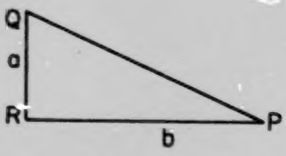
e. Take a point in the exterior of the figure of part (a) and connect it to the two nearest vertices. How does this affect the computation?

f. If you are interested in this problem and would like to pursue it further, you will find it discussed in "The Enjoyment of Mathematics" by Rademacher and Toeplitz and in "Fundamental Concepts of Geometry" by Meserve.

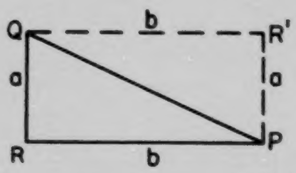
11-2. Areas of Triangles and Quadrilaterals.

Let us now compute some areas, on the basis of our postulates.

Theorem 11-1. The area of a right triangle is half the product of its legs.



$A = \frac{1}{2}\ ab.$



$2A = ab.$

Proof: Given $\Delta\ PQR$, with a right angle at R. Let A be the area of $\Delta\ PQR$. Let R' be the intersection of the parallel to $\overleftrightarrow{PR}$ through Q and the parallel to $\overleftrightarrow{QR}$ through P. Then QR'PR is a rectangle, and $\Delta\ PQR \cong \Delta\ QPR'$. By Postulate 18, this means that the area of $\Delta\ QPR'$ is A. By Postulate 19, the area of the rectangle is $A + A$, because the two triangles intersect only in the segment $\overline{PQ}$. By Postulate 20, the area of the rectangle is ab Therefore

$$2A = ab,$$

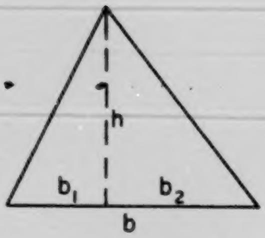
and

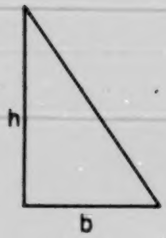
$$A = \frac{1}{2}\ ab,$$

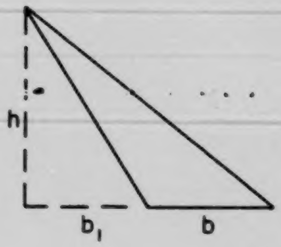
which was to be proved.

From this we can get the formula for the area of any triangle. Once we get this formula, it will include Theorem 11-1 as a special case.

Theorem 11-2. The area of a triangle is half the product of any base and the altitude to that base.







$$A = \frac{1}{2}bh.$$

Proof: Let A be the area of the given triangle. The three figures show the three cases that need to be considered.

(1) If the foot of the altitude is between the two end-points then the altitude divides the given triangle into two right triangles, with bases b_1 and b_2, as indicated. By the preceding theorem, these two triangles have areas $\frac{1}{2}b_1h$ and $\frac{1}{2}b_2h$. By Postulate 19, we have

$$A = \frac{1}{2}b_1h + \frac{1}{2}b_2h.$$

Since $b_1 + b_2 = b$, we have

$$\begin{aligned} A &= \frac{1}{2}(b_1 + b_2)h \\ &= \frac{1}{2}bh, \end{aligned}$$

which was to be proved.

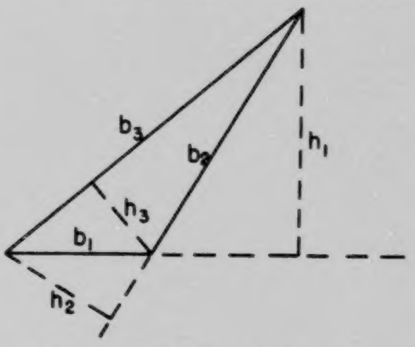
(2) If the foot of the altitude is an end-point of the base, there is nothing left to prove: we already know by the preceding theorem that $A = \frac{1}{2}bh$.

(3) In the third figure, we see the given triangle, with area A, and two right triangles (a big one and a little one.) We have

$$\frac{1}{2}b_1h + A = \frac{1}{2}(b_1 + b)h.$$

The student should supply the reason for this step. Solving algebraically for A, we get $A = \frac{1}{2}bh$, which was to be proved.

Notice that Theorem 11-2 can be applied to any triangle in three ways, because any side can be chosen as the base; we then multiply by the corresponding altitude and divide by 2, to get the area. The figure below shows the three choices for a single triangle.

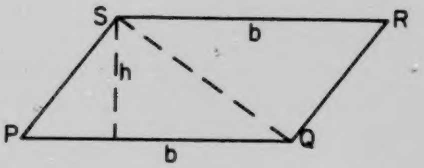


The three formulas $\frac{1}{2}b_1h_1$, $\frac{1}{2}b_2h_2$ and $\frac{1}{2}b_3h_3$ must give the same answer, because all three of them give the right answer for the area of the triangle.

Notice also that once we know how to find the area of a triangle, there is not much left of the area problem for polygonal regions: all we need to do is chop up the polygonal regions into triangular regions (which we know we can do) and then add up the areas of the triangular regions.

For parallelograms and trapezoids this is fairly trivial.

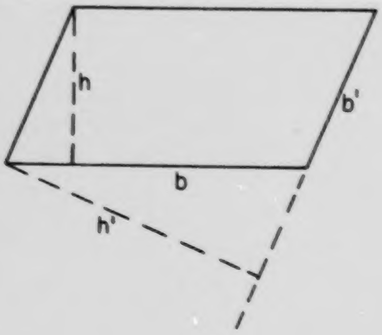
Theorem 11-3. The area of a parallelogram is the product of any base and the corresponding altitude.



$A = bh$

Proof: Draw diagonal $\overline{SQ}$. By Theorem 9-14 $\overline{SQ}$ divides the parallelogram into two congruent triangles. Postulate 18 tells us that congruent triangles have equal area. Now the area of $\triangle PSQ = \frac{1}{2}bh$. Hence the area of parallelogram PQRS = bh, which was to be proved.

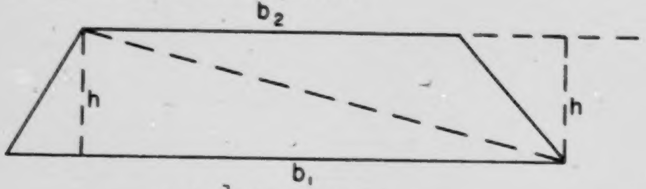
Notice that Theorem 11-3 can be applied to any parallelogram in two ways, because any side can be taken as the base, and can then be multiplied by the corresponding altitude.



In the first case we get $A = bh$, and in the second case we get $A = b'h'$. These two expressions bh and $b'h'$ must give the same answer, because both of them give the right answer for the area of the parallelogram.

The area of a trapezoid can also be obtained by separating it into two triangles.

Theorem 11-4. The area of a trapezoid is half the product of its altitude and the sum of its bases.



$$A = \frac{1}{2}h(b_1 + b_2)$$

Proof: Let A be the area of the trapezoid. Either diagonal divides the trapezoid into two triangles, with areas $\frac{1}{2}b_1h$ and $\frac{1}{2}b_2h$. (The dotted lines on the right indicate why the second triangle has the same altitude h as the first.) By Postulate 19

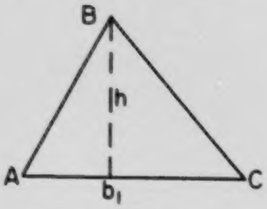
$$A = \frac{1}{2}b_1h + \frac{1}{2}b_2h.$$

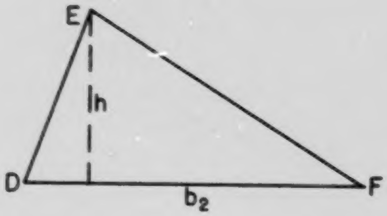
Algebraically, this is equivalent to the formula

$$A = \frac{1}{2}h(b_1 + b_2).$$

The formula for the area of a triangle has two useful consequences, both of which are easy to see:

Theorem 11-5. If two triangles have equal altitudes, then the ratio of their areas is equal to the ratio of their bases.



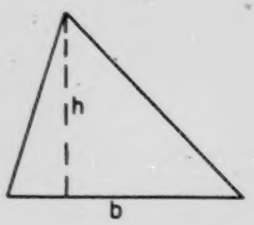


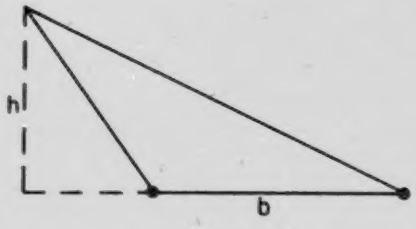
Given: $\triangle$ ABC and $\triangle$ DEF with equal altitudes.

Prove: $$\frac{\text{Area of } \triangle ABC}{\text{Area of } \triangle DEF} = \frac{b_1}{b_2}.$$

This is easy to establish once we have the formula $A = \frac{1}{2}bh$ because it simply means that $\frac{\frac{1}{2}b_1h}{\frac{1}{2}b_2h} = \frac{b_1}{b_2}$, which is true.

Theorem 11-6. If two triangles have equal altitudes and equal bases, then they have equal areas.





The proof of this is clear because the formula $A = \frac{1}{2}bh$ gives the same answer in each case.

Problem Set 11-2

1. In right triangle ABC, with right angle at C, $AC = 7$, $BC = 24$, $AB = 25$.

   a. Find the area of $\triangle ABC$.

   b. Find the altitude to the hypotenuse.

2. The hypotenuse of a right triangle is 30, one leg is 18, and the area of the triangle is 216. Find the length of the altitude to the hypotenuse and the length of the altitude to the given leg.

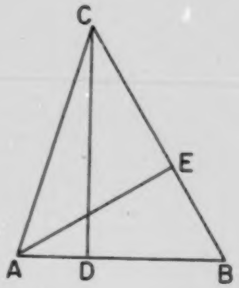
3. In $\triangle ABC$, $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{AE} \perp \overleftrightarrow{BC}$.

   a. If $AB = 8$, $CD = 9$, $AE = 6$, find BC.

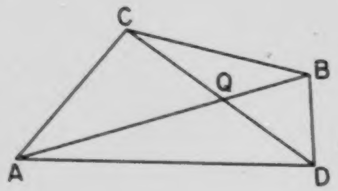
   b. If $AB = 11$, $AE = 5$, $BC = 15$, find CD.

   c. If $CD = 14$, $AE = 10$, $BC = 21$, find AB.

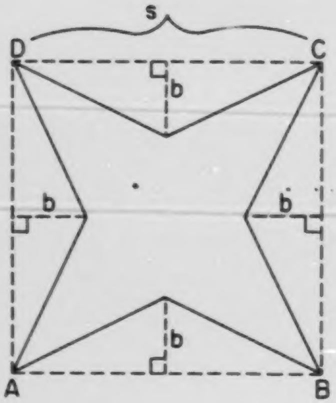
   d. If $AB = c$, $CD = h$, $BC = a$, find AE.



4. In this figure $CQ = QD$. Prove that the Area $\triangle ABC$ = Area $\triangle ABD$.



5. If ABCD is a square, find the area of the star pictured here in terms of s and b. The segments forming the boundary of the star are congruent.



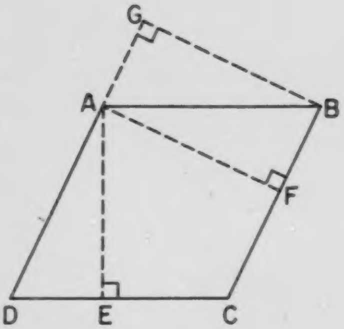
6. In parallelogram ABCD, $\overleftrightarrow{AE} \perp \overleftrightarrow{DC}$, $\overleftrightarrow{AF} \perp \overleftrightarrow{BC}$, and $\overleftrightarrow{BG} \perp \overleftrightarrow{DA}$.

   a. If AE = 7, DC = 12, BC = 14, then AF = ________.

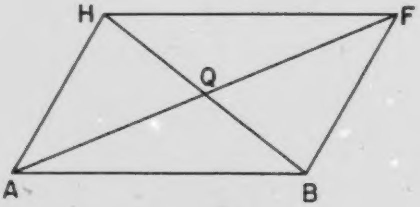
   b. If AE = 10, AB = 18, GB = 15, then AD = ________.

   c. If AF = 6, DC = 14, AE = 8, then AD = ________.

   d. If GB = 16, AD = 20, AF = 16, then AE = ________.



7. Prove that the diagonals of a parallelogram divide it into four triangles which have equal areas.



8. Find the area of trapezoid ABCD,

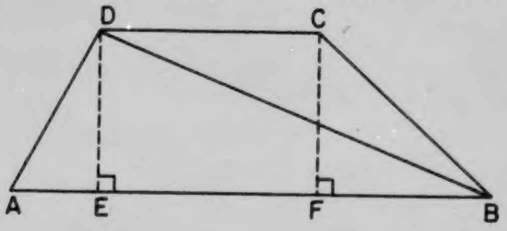
   a. If AB = 12, DC = 6, DE = 4.

   b. If AB = 9 AD = 4, DC = 5, CF = 3.

   c. If AE = 4, FB = 6, DE = 5, DB = 13, DC = 6.

   d. If AB = 27, DE = 7, AE = 3, EF = FB.

   e. If AE = 12, EF = 3, FB = 9, CF = FB.



9. Find the area of a trapezoid if its altitude has length 7 and its median has length 14. (Hint: See Problem 10 of Problem Set 9-6.)

10. A triangle and a parallelogram have equal areas and equal bases. How are their altitudes related?

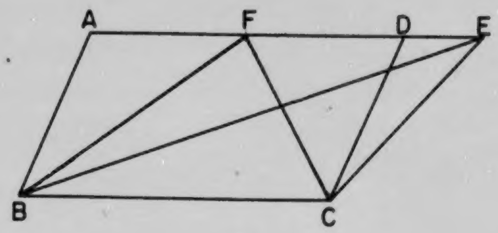
11. Compare the areas of

   a. Parallelogram ABCD and triangle BCE.

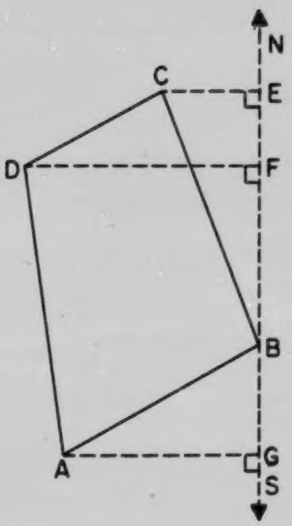
   b. Δ BCF and Δ BCE.

   c. Δ ABF and Δ FCD, if F is the mid-point of $\overline{AD}$.

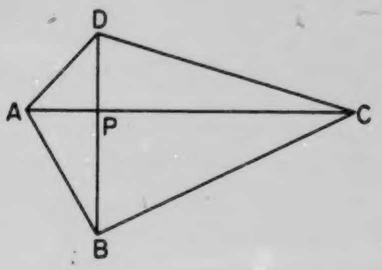
   d. Δ CFD and Δ BCE and parallelogram ABCD, if F is the mid-point of $\overline{AD}$.



12. In surveying the field shown here, a surveyor laid off north-and-south line $\overleftrightarrow{NS}$ through B and then located the east-and-west lines $\overleftrightarrow{CE}$, $\overleftrightarrow{DF}$, $\overleftrightarrow{AG}$. He found that CE = 5 rods, DF = 12 rods, AG = 10 rods, BG = 6 rods, BF = 9 rods, FE = 4 rods. Find the area of the field.



13. Prove the theorem: If quadrilateral ABCD has perpendicular diagonals, its area equals one-half the product of the lengths of the diagonals.

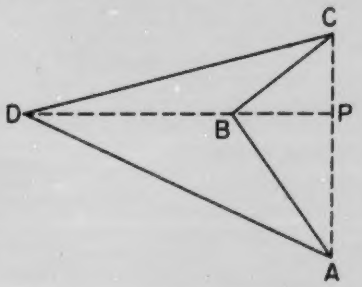


14. Write a corollary to the theorem of Problem 13 relating to the area of a rhombus.

15. The area of a quadrilateral is 126 and the length of one diagonal is 21. If the diagonals are perpendicular, find the length of the other diagonal.

16. The diagonals of a rhombus have lengths of 15 and 20. Find its area. If an altitude of the rhombus is 12, find the length of one side.

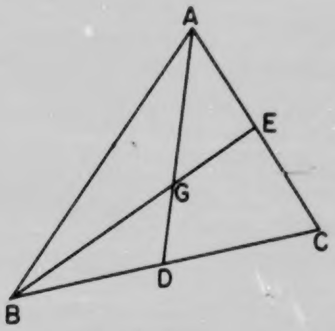
*17. Would the theorem of Problem 13 still be true if the polygonal region ABCD was not convex, as in this figure?



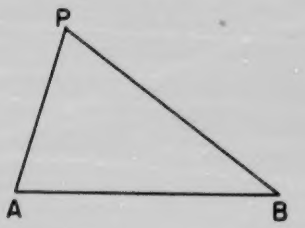
18. Prove that a median of a triangle divides the triangle into two triangles each having an area equal to one-half the area of the original triangle.

19. a. If $\overline{AD}$ and $\overline{BE}$ are two medians of Δ ABC intersecting at G, prove that Area Δ AEG = Area Δ BDG.

b. Determine what part Area Δ BDG is of Area Δ ABC. (Hint: Use other median $\overline{CF}$.)



20. If $\overline{AB}$ is a fixed segment in plane E, what other positions of P in plane E will let the area of Δ ABP remain constant? Describe the location of all possible positions of P in plane E which satisfy the condition. Describe the location of all possible positions of P in space which satisfy the condition.

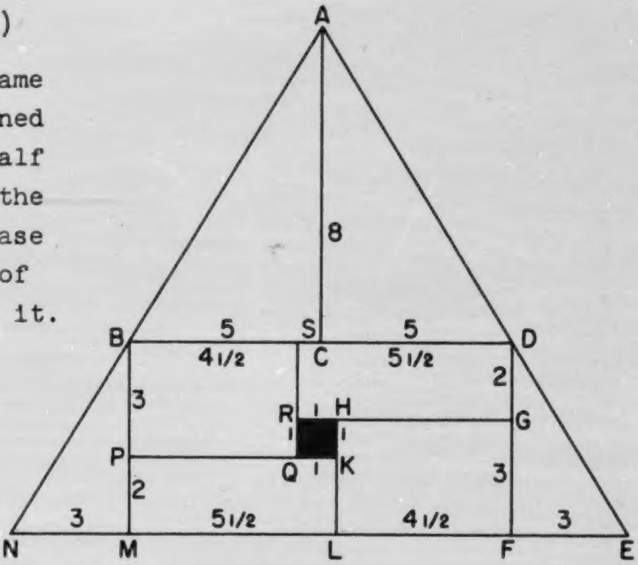


*21. The figure at the right is formed from four right triangles and four rectangles. Notice that there is a square hole one unit on a side.

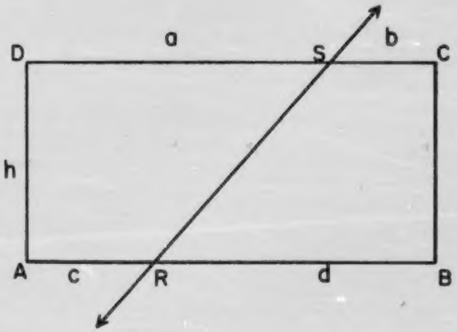
a. Total the areas of the eight parts. (Omit the hole.)

b. Show that the same result is obtained by taking one-half the product of the length of the base and the length of the altitude to it.

c. Explain why the results in (a) and (b) come out the same in spite of the hole.


A
8
B
5
S
5
D
4 1/2
C
5 1/2
2
3
R
1
H
G
1
1
P
Q
1
K
3
2
3
5 1/2
4 1/2
3
N
M
L
F
E

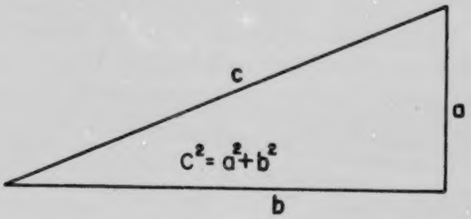
*22. A line cuts a rectangular region into two regions of equal area. Show that it passes through the intersection of the diagonals of the rectangle.


a
b
D
S
C
h
A
c
R
d
B

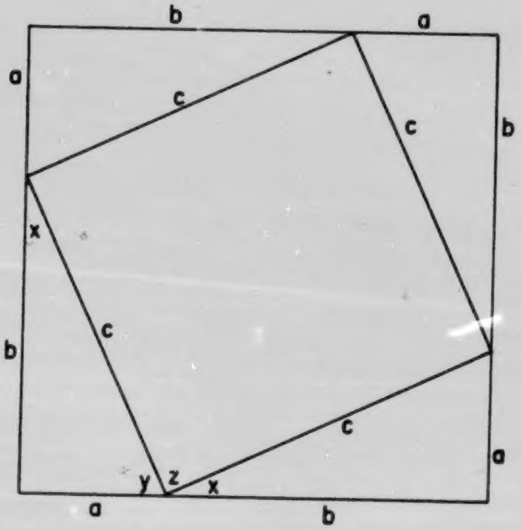
11-3. The Pythagorean Theorem.

Now that we know how to work with areas, the Pythagorean Theorem is actually rather easy to prove.

Theorem 11-7. (The Pythagorean Theorem). In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.



Proof: We take a square for which the length of each side is $a + b$. In this square we draw four right triangles with legs a and b, like this:



then

(1) Each of the four right triangles is congruent to the given triangle by the S.A.S. Postulate. Therefore their hypotenuses have length c, as indicated in the figure above.

(2) The quadrilateral formed by the four hypotenuses is a square. We can show this in the following way:

$\angle z$ is a right angle because $m\angle y + m\angle z + m\angle x = 180$, and $m\angle y + m\angle x = 90$. (The acute angles of a right triangle are complementary). Since all four sides are each equal to c, the quadrilateral is a square.

(3) The area of the large square is equal to the area of the small square, plus the areas of the four congruent right triangles.

Therefore

$$(a + b)^2 = c^2 + 4(\tfrac{1}{2}ab).$$

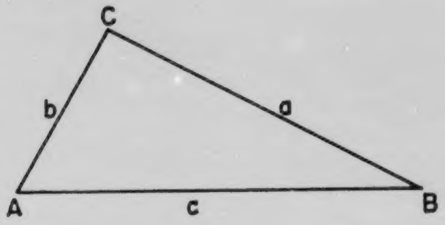
Therefore

$$a^2 + 2ab + b^2 = c^2 + 2ab,$$

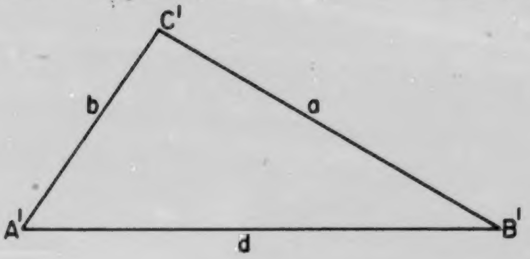
and finally, $a^2 + b^2 = c^2$, which was to be proved.

The converse of the Pythagorean Theorem is also true.

Theorem 11-8. If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle, with a right angle opposite the first side.



Proof: Given ΔABC, as in the figure with $c^2 = a^2 + b^2$. Let $\Delta A'B'C'$ be a right triangle with legs a and b.



Let d be the hypotenuse of the second triangle. By the Pythagorean Theorem,

$$d^2 = a^2 + b^2.$$

Therefore $d^2 = c^2$. Since c and d are both positive, this means that $d = c$. By the S.S.S. Theorem, we have $\triangle A'B'C' \cong \triangle ABC$. Therefore $\angle C \cong \angle C'$. Therefore $\angle C$ is a right angle, which was to be proved.

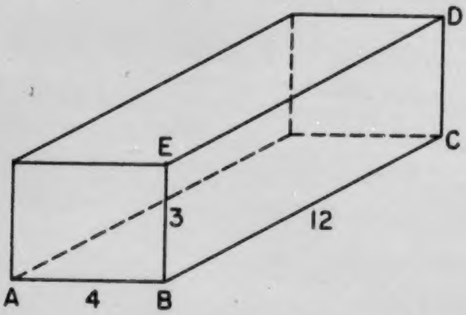
Problem Set 11-3a

1. A man walks due north 10 miles and then due east 3 miles. How far is he from his starting point? ("As the crow flies".)

2. A man walks 7 miles due north, 6 miles due east and then 4 miles north. How far is he from his starting point?

3. A man travels 5 miles north, 2 miles east, 1 mile north, then 4 miles east. How far is he from his starting point?

4. In the rectangular solid indicated in the diagram, find the length of $\overline{AC}$; of $\overline{AD}$.



5. Which of the following sets of numbers could be the lengths of the sides of a right triangle?

   a. 10, 24, 26.

   b. 8, 14, 17.

   c. 7, 24, 25.

   d. 9, 40, 41.

   e. 1.5, 3.6, 3.9.

   f. $1\frac{2}{3}$, $2\frac{2}{3}$, $3\frac{1}{3}$.

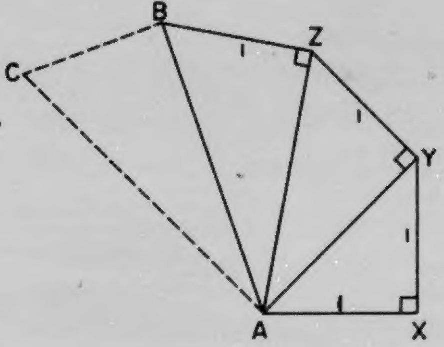
6. a. Show by the converse of the Pythagorean Theorem that integers which represent lengths of sides of right triangles can be found in the following manner.

   Choose any positive integers m and n, where $m > n$. Then $m^2 - n^2$ and $2mn$ will be the lengths of the legs of a right triangle and $m^2 + n^2$ will be the length of its hypotenuse.

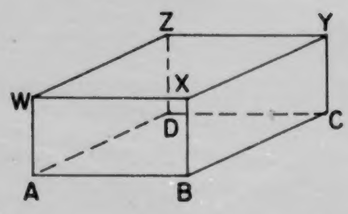
   b. Use the method of part (a) to list integral lengths of sides of right triangles with hypotenuse less than or equal to 25. There are six such triangles.

7. a. With right angles and lengths as marked in the figure, find AY, AZ and AB.

   b. If you continue the pattern established in this figure making $BC = 1$ and $m\angle CBA = 90$, what would be the length of $\overline{AC}$? What would be the length of the next segment from A? You should find an interesting pattern developing.



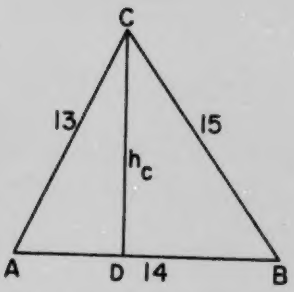
8. In the rectangular solid at the right $AW = 1$, $AB = 2$, $AD = 2$. Find AY.



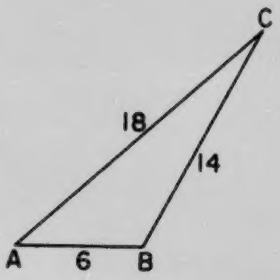
*9. In ΔABC, $AB = 14$, $BC = 15$, $AC = 13$.

a. Find the length of the altitude, h_c, to $\overline{AB}$.

b. Find the length of the altitude, h_a, to $\overline{BC}$.

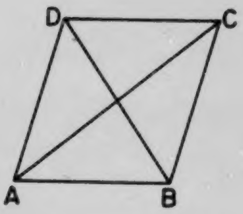


*10. ΔABC has obtuse angle $\angle B$, and $AB = 6$, $BC = 14$, $AC = 18$. Find the length of the altitude, h_c, to $\overleftrightarrow{AB}$.

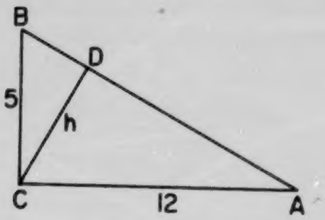


11. One angle of a rhombus has a measure of 60 and one side has length 8. Find the length of each diagonal.

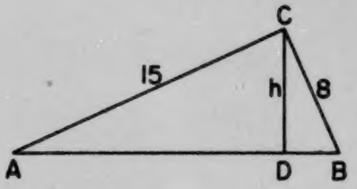
12. In rhombus ABCD, $AC = 6$ and $BD = 4$. Find the length of the perpendicular from any vertex to either opposite side.



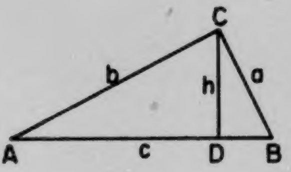
13. In the figure $\overline{BC} \perp \overline{CA}$, $BC = 5$, $CA = 12$, $\overline{CD} \perp \overline{AB}$. Find CD.



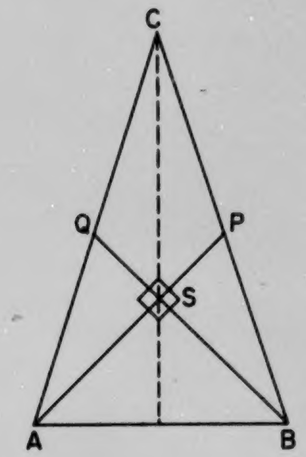
14. The lengths of the legs of right triangle ABC are 15 and 8. Find the length of the hypotenuse. Find the length of the altitude to the hypotenuse.



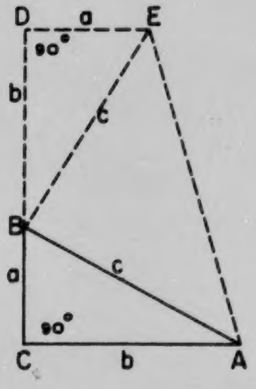
15. If the lengths of the legs of a right triangle ABC are a and b, find the length of the altitude to the hypotenuse.



16. Δ ABC is isosceles with CA = CB. Medians $\overline{AP}$ and $\overline{BQ}$ are perpendicular to each other at S. If SP = n, find the length of each segment and the areas of polygonal regions ASQ, ASB, ABC and QSPC in terms of n. (Do not change radicals to decimals.)

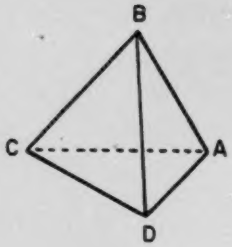


17. A proof of the Pythagorean Theorem making use of the following figure was discovered by General James A. Garfield several years before he became President of the United States. It appeared about 1875 in the "New England Journal of Education." Prove that $a^2 + b^2 = c^2$ by stating algebraically that the area of the trapezoid equals the sum of the areas of the three triangles. You must include proof that $\angle$ EBA is a right angle.



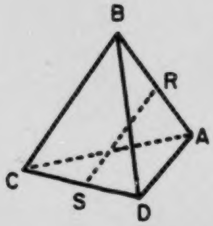
*18. ABCD is a three-dimensional "pyramid-like" solid.

Note that points A, B, C, and D are not coplanar. We are told that BD = BC = BA = AC = CD = DA = 2.

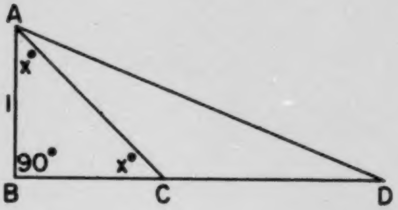


a. R and S are midpoints of $\overline{BA}$ and $\overline{CD}$, respectively. Prove $\overline{RS}$ is perpendicular to both $\overline{BA}$ and $\overline{CD}$.

b. Find the length of RS.

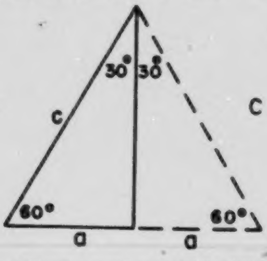


*19. In $\triangle ABD$, $\angle ABD$ is a right angle, AB = BC = 1, AC = CD. Find AD. Find $m\angle ADC$ and $m\angle DAB$.

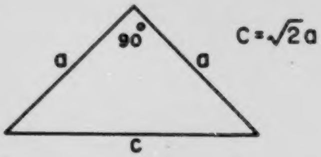


The Pythagorean Theorem also gives us information about the shapes of certain simple triangles. Two very useful relationships are stated in the following two theorems. We give figures which suggest their proofs.

Theorem 11-9. (The 30-60 Triangle Theorem.) The hypotenuse of a right triangle is twice as long as a leg if and only if the measures of the acute angles are 30 and 60.



Theorem 11-10. (The Isosceles Right Triangle Theorem.) A right triangle is isosceles if and only if the hypotenuse is $\sqrt{2}$ times as long as a leg.



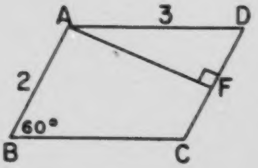
Problem Set 11-3b

1. The lengths of two sides of a triangle are 10 and 14 and the measure of the angle included between these sides is 30. What is the length of the altitude to the side 14? What is the area of the triangle?
2. The measure of the congruent angles of an isosceles triangle are each 30 and the congruent sides each have length 6. How long is the base of the triangle?

3. The measure of one acute angle of a right triangle is double the measure of the other acute angle. If the length of the longer leg is $5\sqrt{3}$, what is the length of the hypotenuse?

4. Show that in any 30° - 60° right triangle with hypotenuse s the length of the side opposite the 60° angle is given by $h = \frac{s}{2}\sqrt{3}$.

5. In parallelogram ABCD, $AB = 2$ and $AD = 3$, $m\angle B = 60$. Find the length of the altitude from A to $\overleftrightarrow{DC}$.



6. If an altitude of an equilateral triangle is 15 inches long, how long is one side of the triangle?

7. In a right triangle with acute angles of 30° and 60°, what is the ratio of the shortest side to the hypotenuse? Of the hypotenuse to the shortest side? Of the shortest side to the side opposite the 60° angle? Of the side opposite the 60° angle to the shortest side? Of the side opposite the 60° angle to the hypotenuse? Of the hypotenuse to the side opposite the 60° angle? Are these ratios the same for every 30° - 60° right triangle? If you have done this problem carefully, you should find the results very helpful in many of the following problems.

8. What is the area of the isosceles triangle whose congruent sides have lengths of 20 inches each and whose base angles have measures of:

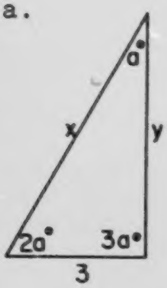
a. 30? b. 45? c. 60?

9. What is the area of the isosceles triangle whose base has a length of 24 inches and whose base angles each have measures of:

a. 45? b. 30? c. 60?

10. Use the information given in the figures to determine the numerical values called for below:

a.



a =

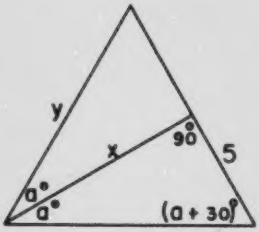
2a =

3a =

x =

y =

b.

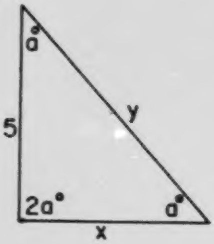

y
x
90°
5
a°
a°
(a + 30)°

a =

x =

y =

c.



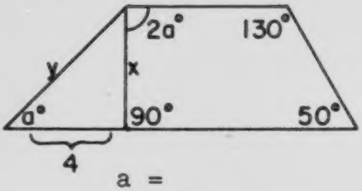
a =

2a =

x =

y =

d.

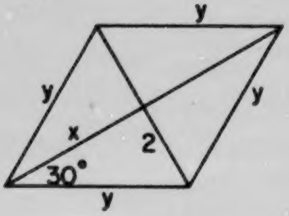

2a°
130°
y
x
a°
90°
50°
4

a =

x =

y =

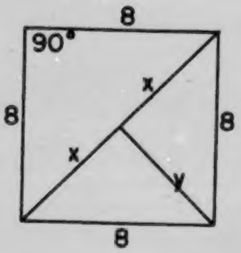
e.



x =

y =

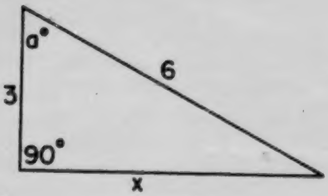
f.



x =

y =

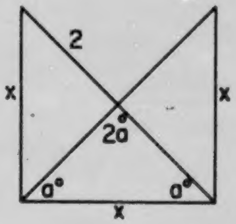
g.



a =

x =

h.

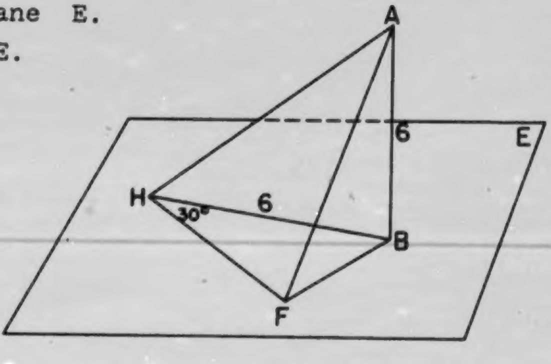


a =

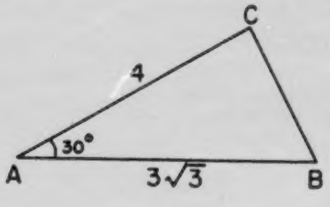
x =

11. In this figure $\overline{AB} \perp$ plane E. ΔBFH lies in plane E. $\overline{HF} \perp \overline{FB}$. AB = BH = 6. $m\angle FHB = 30$.

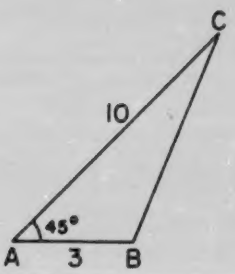
Give the measures of as many other segments and angles of the figure as you can determine.



*12. In ΔABC, $m\angle A = 30$, AC = 4, AB = $3\sqrt{3}$. Find BC. Is $\angle C$ a right angle?

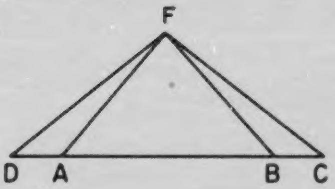

C
4
30°
A
$3\sqrt{3}$
B

*13. In ΔABC as shown in the figure, find BC. (Hint: Draw the altitude from C.)

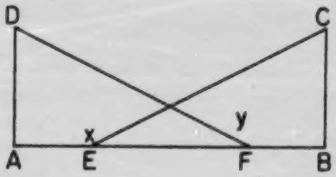


14. The base of an isosceles triangle is 20 inches and a leg is 26 inches. Find the area.

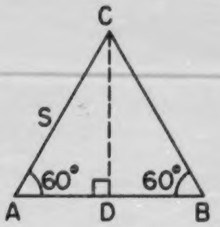
15. In this figure $FD = FC$, $DB = CA$, $\overline{DF} \perp \overline{FB}$, and $\overline{CF} \perp \overline{FA}$. Prove ΔFAB is isosceles.



16. $\overline{DA}$ and $\overline{CB}$ are both perpendicular to $\overline{AB}$ in this figure. $AE = FB$ and $DF = CE$. Prove $\angle x \cong \angle y$.



17. Prove the theorem: The area of an equilateral triangle with side s is given by Area $= \frac{s^2}{4}\sqrt{3}$.



18. Find the area of an equilateral triangle having the length of a side equal to:

a. 2.

b. 8.

c. $\sqrt{3}$.

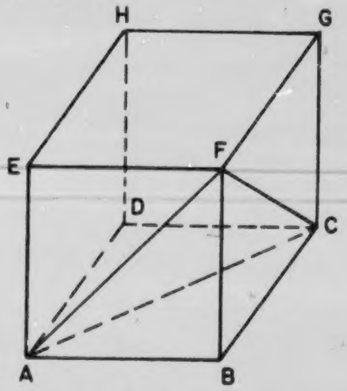
d. 7.

19. The area of an equilateral triangle is $9\sqrt{3}$. Find its side and its altitude.

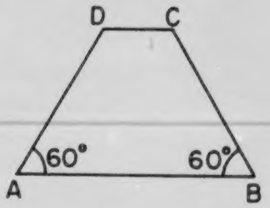
20. The area of an equilateral triangle is $16\sqrt{3}$. Determine its side and its altitude.

21. A square whose area is 81 has its perimeter of length equal to the length of the perimeter of an equilateral triangle. Find the area of the equilateral triangle.

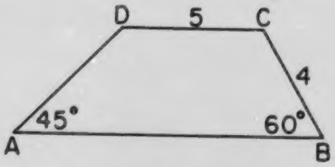
22. This figure represents a cube. The plane determined by points A, C and F is shown. If AB is 9 inches, how long is $\overline{AC}$? What is the measure of $\angle$ FAC? What is the area of Δ FAC?



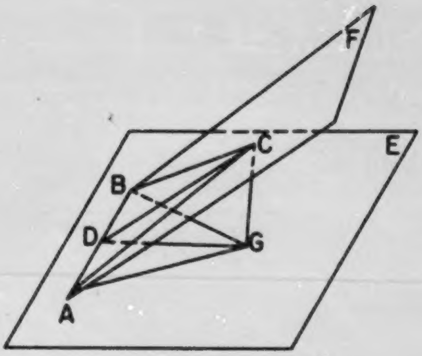
23. In trapezoid ABCD, base angles of 60° include a base of length 12. The non-parallel side $\overline{AD}$ has length 8. Find the area of the trapezoid.



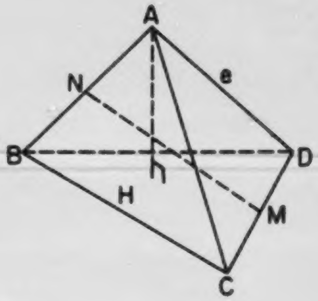
24. Find the area of the trapezoid.



*25. In the figure, plane E and plane F intersect in $\overleftrightarrow{AB}$, forming dihedral angle $\angle$ F-AB-E. $\overline{CG} \perp$ plane E, $\overline{DG} \perp \overline{AB}$, and $\overline{CD} \perp \overline{AB}$. D is the mid-point of $\overline{AB}$. $\overline{BC} \cong \overline{AC}$. If $AB = 4\sqrt{6}$, $AG = 6$, $m\angle CBG = 45$, and $m\angle CAG = 45$, find CG and $m\angle$ F-AB-E.



*26. Figure ABCD is a regular tetrahedron (its faces are equilateral). Let any edge be e. $\overline{NM} \perp \overline{AB}$ and $\overline{NM} \perp \overline{DC}$.

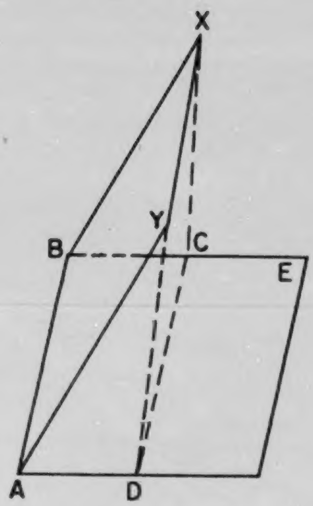


a. Show that the length of a bi-median, that is, the segment, $\overline{NM}$, joining the mid-points of opposite edges, is $\frac{\sqrt{2}}{2}$ e.

(Hint: Draw $\overline{AM}$.)

b. Show that the length of the altitude, $\overline{AH}$, of the tetrahedron is $\frac{\sqrt{6}}{3}$ e. (Hint: Draw $\overline{HC}$ and $\overline{HD}$. Does H lie on $\overline{BM}$? Recall that the medians of a triangle are concurrent at a point $\frac{2}{3}$ of the distance from each vertex.)

27. ABXY is a square. AB = 6. m∠ X-AB-E = 60. Rectangle ABCD is the projection of square ABXY on plane E. What is the area of rectangle ABCD?

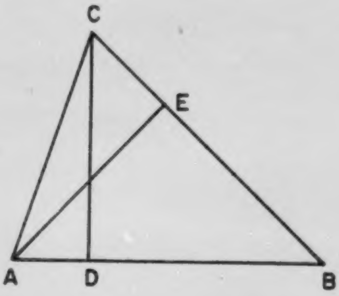


*28. Given any two rectangles anywhere in a plane, how can a single line be drawn which will separate each rectangular region into two regions of equal area?

Review Problems

1. If the side of one square is double the side of another square, then the area of the first square is ______ times the area of the second square.

2. In Δ ABC, $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{AE} \perp \overleftrightarrow{BC}$, AB = 8, CD = 9 and AE = 6. Find BC.



3. A man walks 5 miles north, then 2 miles east, then 1 mile north, then 6 miles east. How far will he be from his starting point?

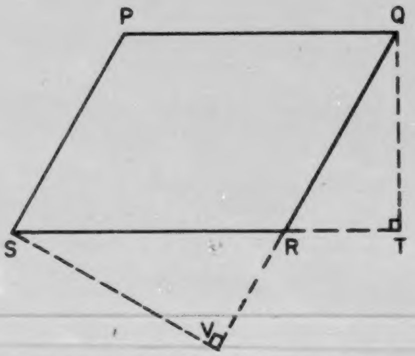
4. If the diagonal of a square is 15 feet long, how long is each side?

5. Find the area of an isosceles triangle in which the base is 12 and each congruent side is 10.

6. In the figure, PQRS is a parallelogram, $\overleftrightarrow{QT} \perp \overleftrightarrow{SR}$, and $\overleftrightarrow{SV} \perp \overleftrightarrow{QR}$.

   a. If $SV = 7$ and $PS = 5$, find the area of PQRS.

   b. If $SV = 8$, $QT = 4$ and $SR = 10$, find QR.

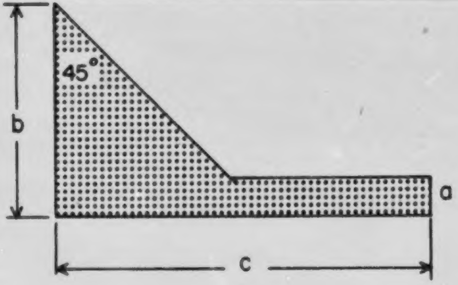


7. In an equilateral triangle the length of the altitude is 6 inches. What is the length of each side?

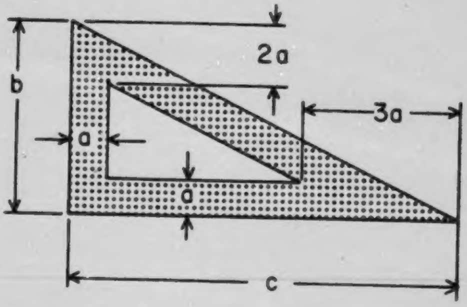
8. The side of a rhombus is 13 and one of its diagonals is 24. Find its area.

9. In $\triangle$ ABC base $AB = 12$, median $CD = 8$, and $m\angle ADC = 30$. The area of $\triangle$ ABC is _______.

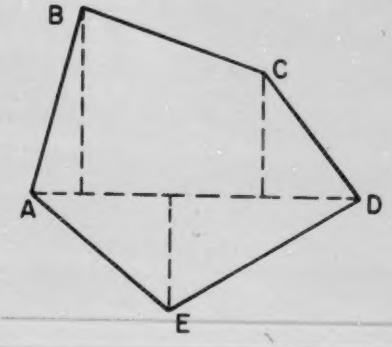
10. Derive a formula for the area of the figure at the right in terms of the indicated lengths.



11. Find the area of the shaded region of the Figure at the right.

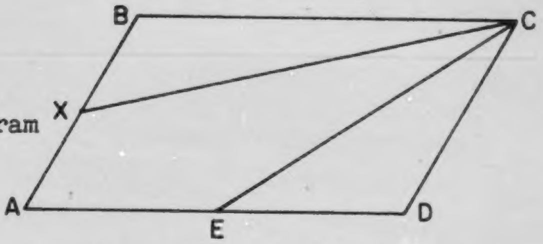


12. Diagonal $\overline{AD}$ of the pentagon ABCDE shown is 44 and the perpendiculars from B, C, and E are 24, 16, and 15 respectively. AB = 25 and CD = 20. What is the area of the pentagon?



13. Given: Parallelogram ABCD with X and E midpoints of $\overline{AB}$ and $\overline{AD}$ respectively.

To prove: Area of region AECX = $\frac{1}{2}$ area parallelogram ABCD.



14. Prove that the area of an isosceles right triangle is equal to one fourth the area of a square having the hypotenuse of the triangle as a side.

*15. An equilateral triangle has one side in a given plane. The plane of the triangle is inclined to the given plane at an angle of 60°. What is the ratio of the area of the triangle to the area of its projection on the plane?

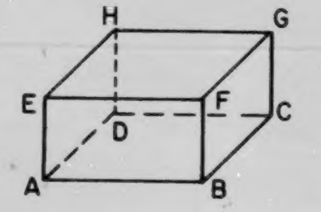
*16. Explain how to divide a trapezoid into two parts that have equal areas by a line through a vertex.

*17. Find the length of the diagonal of a cube whose edge is 6 units long.

*18. In this rectangular solid $AE = 5$, $AB = 10$ and $AD = 10$.

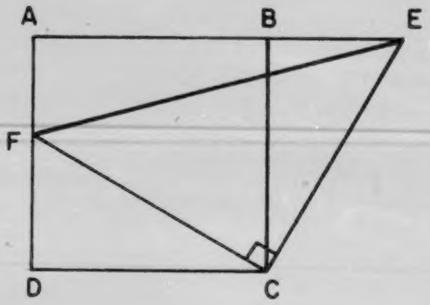
a. Find AC.

b. Find AG.

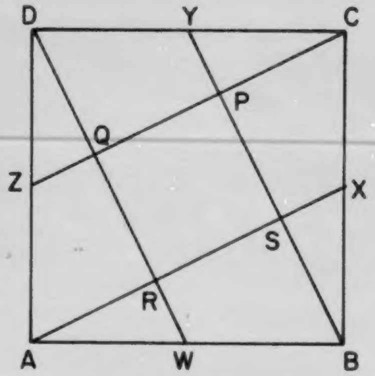


*19. Given: Square ABCD with points E and F as shown, so that $\overline{EC} \perp \overline{FC}$.
Area ABCD = 256 sq. ft.
Area of ΔCEF = 200 sq. ft.

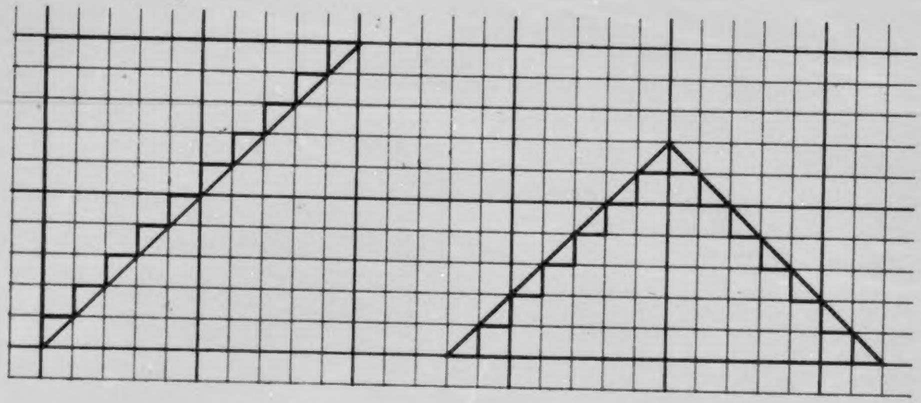
Find BE.



*20. If W, X, Y and Z are mid-points of sides of square ABCD, as shown in the figure, compare the area of this square with that of square RSPQ.



*21. The figure shows two isosceles right triangles. The first of these has a horizontal side of length 10 units and the second has a horizontal hypotenuse of length 14 units.



a. Draw two such triangles on graph paper. Cut out the second one and place it on the first to show that their areas are apparently equal.

b. In the first figure count the number of small squares and the number of small half squares (right isosceles triangles). Use these numbers to compute the area.

c. Do the same for the second figure.

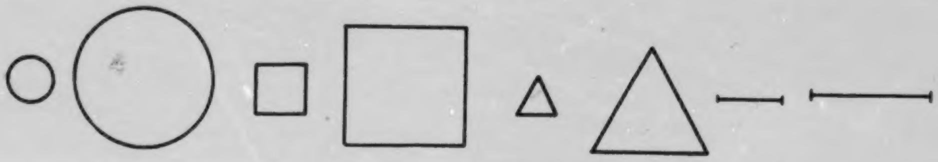
d. Explain the discrepancy.

Chapter 12

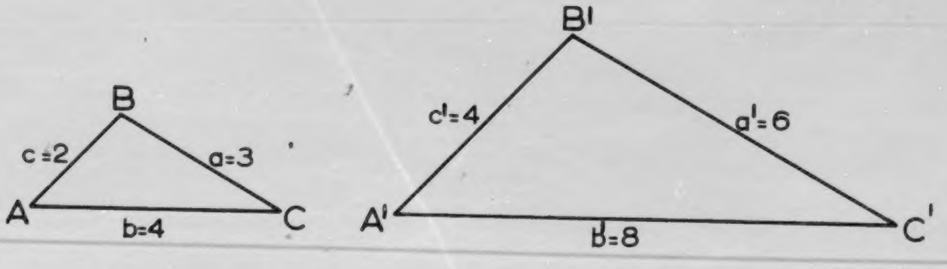
SIMILARITY

12-1. The Idea of a Similarity.

Proportionality. Roughly speaking, two geometric figures are similar if they have exactly the same shape, but not necessarily the same size. For example, any two circles are similar; any two squares are similar; any two equilateral triangles are similar; and any two segments are similar.



Below are two triangles, with the lengths of the sides as indicated:



These figures stand in a very special kind of relation to each other. One way to describe this relation, speaking very roughly, is to say that the triangle on the left can be "stretched", or the one on the right can be "shrunk", so as to match up with the other triangle, by the correspondence

$$ABC \leftrightarrow A'B'C'.$$

Of course, this correspondence is not a congruence, because each side of the right-hand triangle is twice as long as the corresponding side of the other. Correspondences of this type are called similarities. The exact definition of a similarity will be given later in this chapter.

Notice that the lengths of the sides of our two triangles form two sequences of positive numbers, a, b, c and a', b', c', standing in a very special relation: each number in the second sequence is exactly twice the corresponding number in the first sequence; or, said another way, each number in the first sequence is exactly half the corresponding number in the second sequence. Thus

$$\begin{aligned} a' &= 2a, & & & a &= \tfrac{1}{2}a', \\ b' &= 2b, & &\text{or} & b &= \tfrac{1}{2}b', \\ c' &= 2c; & & & c &= \tfrac{1}{2}c'. \end{aligned}$$

Another way of putting this is to write

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = 2, \quad \text{or} \quad \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{1}{2}$$

Sequences of positive numbers which are related in this way are called proportional.

Definition: Two sequences of numbers, a, b, c, ... and p, q, r, ..., none of which is zero, are proportional if

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r} = \ldots \quad \text{or} \quad \frac{p}{a} = \frac{q}{b} = \frac{r}{c} = \ldots.$$

The simplest proportionalities are those involving only four numbers, and these have special properties that are worth noting. We list some of them for later reference.

Algebraic Properties of a Simple Proportion.

If $$\frac{a}{b} = \frac{c}{d},$$

with a, b, c, d all different from zero,

then

(1) $$ad = bc,$$

(2) $$\frac{a}{c} = \frac{b}{d},$$

(3) $$\frac{a+b}{b} = \frac{c+d}{d},$$

(4) $$\frac{a-b}{b} = \frac{c-d}{d},$$

Proof: Taking the original equation $\frac{a}{b} = \frac{c}{d}$,

(1) Multiply both sides by bd to get $ad = bc$;

(2) Multiply both sides by $\frac{b}{c}$ to get $\frac{a}{c} = \frac{b}{d}$;

(3) Add 1 to both sides to get $\frac{a+b}{b} = \frac{c+d}{d}$;

(4) Subtract 1 from both sides to get $\frac{a-b}{b} = \frac{c-d}{d}$.

Other relations can be derived, but these are the most useful.

Definition: If a, b, c are positive numbers and $\frac{a}{b} = \frac{b}{c}$, then b is the geometric mean between a and c.

From Property (1) above, it follows that the geometric mean between a and c is $\sqrt{ac}$.

Problem Set 12-1

1. Complete each statement:

a. If $\frac{a}{b} = \frac{3}{7}$ then $7a =$ ______.

b. If $\frac{x}{3} = \frac{1}{4}$ then $4x =$ ______.

c. If $\frac{6}{5} = \frac{4}{y}$ then $6y =$ ______.

2. In each of the following proportionalities, find x.

a. $\frac{x}{2} = \frac{3}{4}$. c. $\frac{5}{4} = \frac{x}{13}$.

b. $\frac{5}{x} = \frac{4}{7}$. d. $\frac{2}{3} = \frac{11}{x}$.

3. Complete each statement:

a. If $3a = 2x$, then $\frac{a}{x} =$ ——, and $\frac{a}{2} =$ ——.

b. If $5 \cdot 3 = 4m$, then $\frac{4}{3} =$ ——, and $\frac{m}{3} =$ ——.

c. If $7b = 4a$, then $\frac{a}{b} =$ ——, and $\frac{b}{a} =$ ——.

d. If $5 \cdot 9 = 6x$, then $\frac{x}{5} =$ ——, and $\frac{5}{x} =$ ——.

4. In each of the following proportionalities, express the number a in terms of the numbers b, c and d.

a. $\frac{2a}{3b} = \frac{4c}{5d}$. c. $\frac{3b}{4c} = \frac{5a}{7d}$.

b. $\frac{2b}{5a} = \frac{7c}{11d}$. d. $\frac{b}{2c} = \frac{6d}{5a}$.

*5. Complete each statement:

a. If $\frac{a}{b} = \frac{3}{1}$, then $\frac{a + b}{b} =$ ——, and $\frac{a - b}{b} =$ ——.

b. If $\frac{x}{3} = \frac{y}{2}$, then $\frac{y + 2}{2} =$ ——, and $\frac{y - 2}{2} =$ ——.

c. If $\frac{a + c}{c} = \frac{11}{7}$, then $\frac{a}{c} =$ ——, and $\frac{a - c}{c} =$ ——.

d. If $\frac{a}{b} = \frac{5}{3}$, then $\frac{b + a}{a} =$ ——, and $\frac{b - a}{a} =$ ——.

6. Here are three sequences of numbers. Are any two pairs of sequences proportional?

a. 3, 7, 12.

b. 9, 21, 36.

c. $\frac{5}{2}$, $\frac{35}{6}$, 10.

One can tell at a glance that the sequences a and b are proportional since each number in b is 3 times the corresponding number in a. The comparison of a and c is not such a simple matter. An efficient way to make such a comparison might be to change each to a new proportional sequence beginning with 1, that is,

a. 1, $\frac{7}{3}$, 4.

b. 1, $\frac{7}{3}$, —.

c. 1, —, —.

7. In the following list of sequences of numbers, which pairs of sequences are proportional? Make a complete list of these pairs of sequences.

a. 5, 7, 9.

b. 1, 2, 3.

c. 9, 7, 17.

d. $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$.

e. 18, 14, 34.

f. $\frac{1}{3}$, $\frac{2}{3}$, 1.

g. 27, 21, 51.

h. 15, 30, 45.

i. 10, 14, 18.

8. If $\frac{w}{40} = \frac{v}{50} = \frac{20}{1}$, what are the values of w and v?

9. If $\frac{3}{x} = \frac{4}{y} = \frac{11}{z} = \frac{4}{1}$, what are the values of x, y and z?

10. Which of the following are correct for all values of the letters involved assuming that no number in any sequence shown is zero?

a. $\frac{3}{13} = \frac{4}{14}$.

b. $\frac{j}{10j} = \frac{k}{10k}$.

c. $\frac{r}{r^2} = \frac{s}{rs} = \frac{t}{st}$.

d. $\frac{a + b}{a^2 + b^2} = \frac{1}{a + b}$.

e. $\frac{x}{x^2} = \frac{y}{y^2} = \frac{z}{z^2} = \frac{w}{w^2}$.

f. $\frac{1}{c + d} = \frac{c - d}{c^2 - d^2}$.

11. If $\frac{16}{40} = \frac{p}{45} = \frac{q}{60} = \frac{28}{t}$, what are the values of p, q and t?

12. The geometric mean of two positive numbers a and c is $b = \sqrt{ac}$. The arithmetic mean of a and c is $d = \frac{a + c}{2}$. Find the geometric mean and the arithmetic mean of the following pairs:

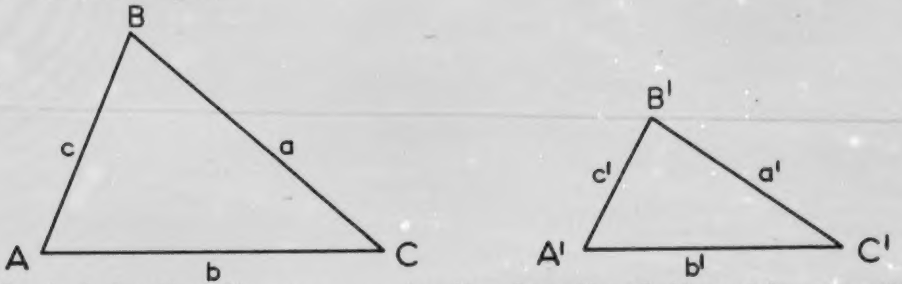
a.	4 and 9.	d.	2 and 24.
b.	6 and 12.	e.	2 and 3.
c.	8 and 10.		

12-2. Similarities between Triangles.

We can now state the definition of a similarity between two triangles. Suppose we have given a correspondence

$$ABC \longleftrightarrow A'B'C'$$

between two triangles



As indicated in the figure, a is the length of the side opposite A, b is the length of the side opposite B, and so on. If corresponding angles are congruent, and

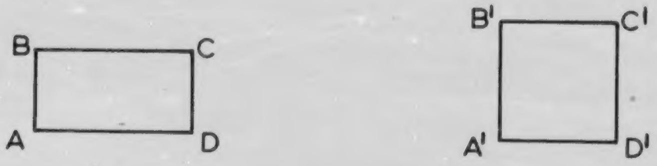
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'},$$

then the correspondence $ABC \longleftrightarrow A'B'C'$ is a similarity, and we write

$$\triangle ABC \sim \triangle A'B'C'.$$

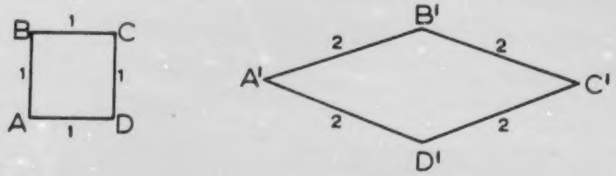
Definition: Given a correspondence between the vertices of two triangles. If corresponding angles are congruent and the corresponding sides are proportional, then the correspondence is a similarity, and the triangles are said to be similar.

Notice that this definition requires two things: (1) corresponding angles must be congruent, and (2) corresponding sides must be proportional. In putting both of these requirements into the definition, we are making sure that the definition may be applied to polygonal figures of more than three sides. To see what the possible troubles might be, if we used only one of our two requirements, let us look at the situation for quadrilaterals.



First consider the correspondence ABCD↔A'B'C'D', between the two rectangles in the figure. Corresponding angles are congruent, because all of the angles are right angles, but the two rectangles don't have the same shape, by any means.

Now consider a square and a rhombus, with edges of length 1 and 2, like this:

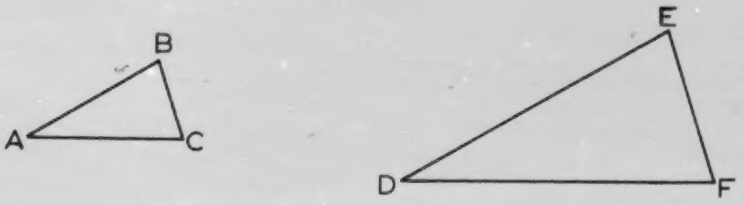


Under the correspondence ABCD↔A'B'C'D', corresponding sides are proportional, but the shapes are quite different.

We shall see later that for the case of correspondences between triangles, if either one of our conditions holds, then so does the other. That is, if corresponding angles are congruent, then corresponding sides are proportional; and conversely, if corresponding sides are proportional, then corresponding angles are congruent. These facts are given in the A.A.A. Similarity Theorem and the S.S.S. Similarity Theorem, which will be proved later in this chapter.

Problem Set 12-2

1. Given a similarity $\triangle ABC \sim \triangle DEF$,



write down the proportionality between corresponding sides, using the notation AB, AC, and so on. Then:

a. Express AB in terms of AC, DE and DF.

b. Express BC in terms of AB, DE and EF.

c. Express AC in terms of BC, EF and DF.

d. Express AB in terms of BC, DE and EF.

e. Express BC in terms of AC, EF and DF.

f. Express AC in terms of AB, DE and DF.

2. Below are listed five sets of 3 numbers. Point out which pairs of sets of numbers (not necessarily in the order given) might be lengths of sides of similar triangles. Write out the equal ratios in each case. For example, a, b; $\frac{3}{6} = \frac{4}{8} = \frac{6}{12}$.

   a. 3, 4, 6.
   b. 8, 6, 12.
   c. 3, 4, 9.
   d. 9, 12, 18.
   e. 2, $4\frac{1}{2}$, 4.

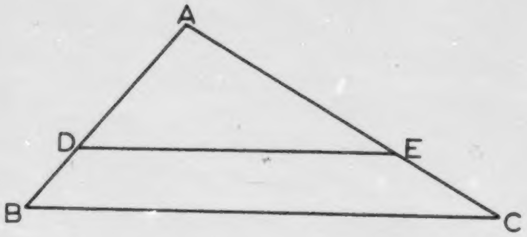
3. Two prints of a negative are made, one a contact print and one enlarged. In the contact print an object has a length of 2 inches and a height of 1.6 inches. In the enlarged print the same object has a length of 7.5 inches. Find its height in the enlargement.

4. If $\Delta ABC \cong \Delta A'B'C'$, does it follow that $\Delta ABC \sim \Delta A'B'C'$? Why or why not?

°5. Prove: The triangle whose vertices are the mid-points of the sides of a given triangle is similar to the given triangle.

12-3. The Basic Similarity Theorems.

Consider a triangle ΔABC. Let D and E be different points on the sides $\overline{AB}$ and $\overline{AC}$, and suppose that $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ are parallel.



It looks as if the correspondence

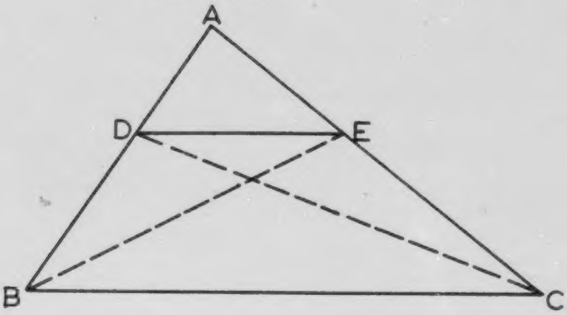
$$ABC \leftrightarrow ADE$$

ought to be a similarity, and it is, as we shall presently see. We prepare the way with a series of theorems.

Theorem 12-1. (The Basic Proportionality Theorem.) If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off segments which are proportional to these sides.

Restatement: In ΔABC let D and E be points of $\overline{AB}$ and $\overline{AC}$ such that $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$. Then

$$\frac{AB}{AD} = \frac{AC}{AE}.$$



Proof: (1) In ΔADE and ΔBDE think of $\overline{AD}$ and $\overline{BD}$ as the bases and the altitude from E to $\overleftrightarrow{AB}$ as their common altitude. Then by Theorem 11-5,

$$\frac{\text{area } \Delta BDE}{\text{area } \Delta ADE} = \frac{BD}{AD}.$$

(2) In ΔAED and ΔCED think of $\overline{AE}$ and $\overline{CE}$ as the bases and the altitude from D to $\overleftrightarrow{AC}$ as their common altitude. Then by Theorem 11-5,

$$\frac{\text{area } \Delta CDE}{\text{area } \Delta ADE} = \frac{CE}{AE}.$$

(3) ΔBDE and ΔCDE have the same base, $\overline{DE}$, and congruent altitudes, since the lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ are parallel. Hence by Theorem 11-6,

$$\text{area } \Delta BDE = \text{area } \Delta CDE.$$

(4) It follows from (1), (2) and (3) that

$$\frac{BD}{AD} = \frac{CE}{AE}$$

Applying Algebraic Property (3), from Section 12-1,

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

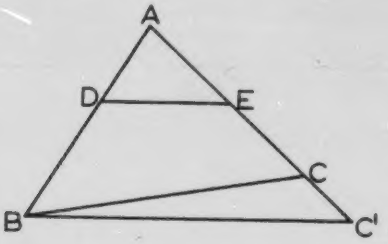
The converse of Theorem 12-1 is also true (and is easier to prove). That is, we have:

Theorem 12-2. If a line intersects two sides of a triangle, and cuts off segments proportional to these two sides, then it is parallel to the third side.

Restatement: Let $\triangle ABC$ be a triangle. Let D be a point between A and B, and let E be a point between A and C. If

$$\frac{AB}{AD} = \frac{AC}{AE},$$

then $\overleftrightarrow{BC}$ and $\overleftrightarrow{DE}$ are parallel.



Proof: Let $\overleftrightarrow{BC'}$ be the line through B, parallel to $\overleftrightarrow{DE}$, and intersecting $\overleftrightarrow{AE}$ in C'. By Theorem 12-1,

$$\frac{AB}{AD} = \frac{AC'}{AE},$$

so that

$$AC' = AE \cdot \frac{AB}{AD}.$$

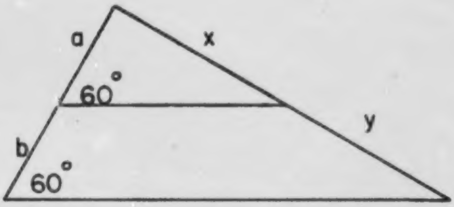
But the equation given in the hypothesis of the theorem means that

$$AC = AE \cdot \frac{AB}{AD}.$$

Therefore AC' = AC. Therefore C' = C, and $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{DE}$, which was to be proved.

Problem Set 12-3a

1. In this figure the lengths of segments are a, b, x and y as shown.

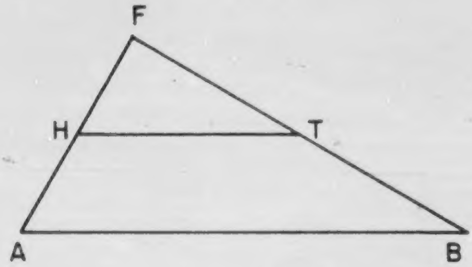


$\frac{a+b}{a} = \frac{\underline{\quad\quad}}{x}$. $\frac{a}{b} = \underline{\quad\quad}$.

$\frac{a+b}{b} = \frac{x + \underline{\quad}}{\underline{\quad\quad}}$. $\frac{a}{x} = \underline{\quad\quad}$.

$\frac{a+b}{x+y} = \frac{\underline{\quad\quad}}{x}$. $\frac{x+y}{a+b} = \frac{y}{\underline{\quad\quad}}$.

2. In this figure if $\overline{HT} \parallel \overline{AB}$,

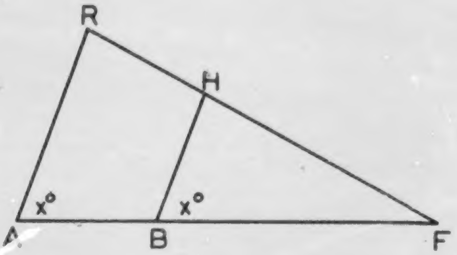


$\frac{\overline{FA}}{\overline{FH}} = \underline{\quad\quad}$. $\frac{\overline{TB}}{\overline{FT}} = \underline{\quad\quad}$.

$\frac{\overline{FA}}{\overline{HA}} = \underline{\quad\quad}$. $\frac{\overline{FT}}{\overline{FH}} = \underline{\quad\quad}$.

$\frac{\overline{FH}}{\overline{HA}} = \underline{\quad\quad}$. $\frac{\overline{BT}}{\overline{AH}} = \underline{\quad\quad}$.

3. In the figure,

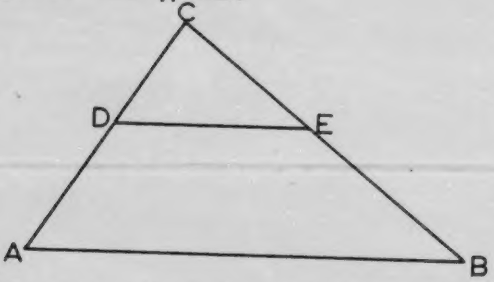


a. If RH = 4, HF = 7, BF = 10, then AB = ____.

b. If RH = 6, HF = 10, AB = 3, then BF = ____.

c. If RH = 5, RF = 20, AF = 18, then BF = ____.

4. In the figure, $\overline{DE} \parallel \overline{AB}$.



a. If AC = 12, CD = 4, CE = 8, find BC.

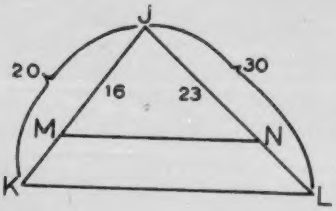
b. If AD = 6, BE = 10, CD = 4, find CE.

c. If BC = 22, EB = 6, CD = 8, find AC.

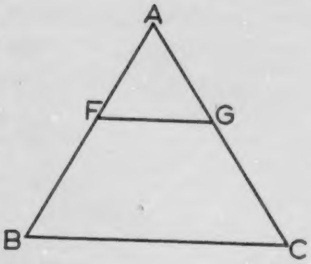
d. If AD = 5, CD = 7, BC = 18, find BE.

e. If AC = 15, CE = 6, BC = 18, find AD.

5. In the figure let the segments have measures as indicated. Can $\overline{MN} \parallel \overline{KL}$? Justify your answer.



6. Which of the following sets of data make $\overline{FG} \parallel \overline{BC}$?



a. AB = 14, AF = 6, AC = 7, AG = 3.

b. AB = 12, FB = 3, AC = 8, AG = 6.

c. AF = 6, FB = 5, AG = 9, GC = 8.

d. AC = 21, GC = 9, AB = 14, AF = 5.

e. AB = 24, AC = 6, AF = 8, GC = 4.

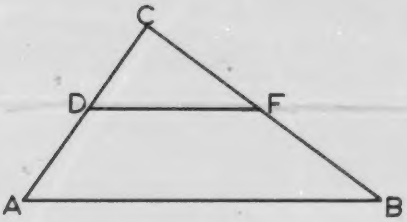
7. If, in the figure, $\overline{DF} \parallel \overline{AB}$, prove

a. $\frac{DA}{CD} = \frac{FB}{CF}$.

Hint: Use Theorem 12-1 and subtract 1 from each fraction.

b. $\frac{CA}{DA} = \frac{CB}{FB}$.

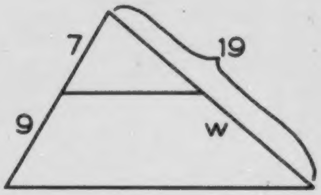
c. $\frac{CA}{CB} = \frac{CD}{CF}$.



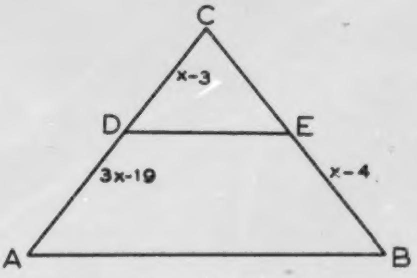
8. Given the figure, one person handled the problem of finding w in this way:

$$\frac{7}{9} = \frac{19 - w}{w}.$$

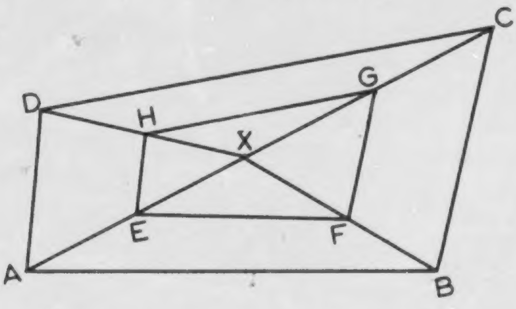
Propose a more convenient equation. Do you get the same result?



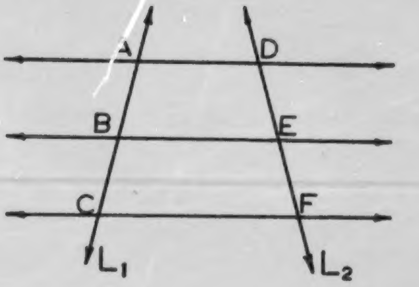
9. Place conditions upon x such that $\overline{DE} \parallel \overline{AB}$, given that $CD = x - 3$, $DA = 3x - 19$, $CE = 4$, and $EB = x - 4$.



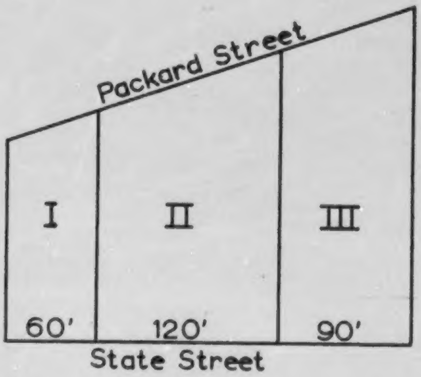
10. In this figure if $\overline{EF} \parallel \overline{AB}$, $\overline{FG} \parallel \overline{BC}$, and $\overline{GH} \parallel \overline{DC}$, prove $\overline{HE} \parallel \overline{DA}$. Must the figure be planar?



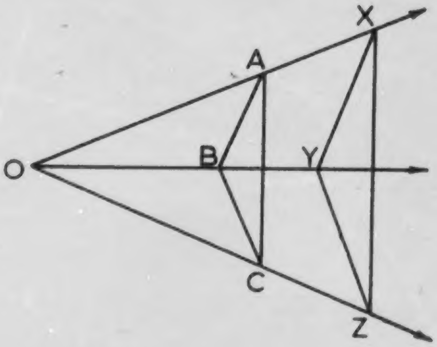
11. Prove: If three or more parallels are cut by two transversals, the intercepted segments on the two transversals are proportional.
Restatement: If the lines L_1 and L_2 are transversals of the parallel lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$, then $\frac{AB}{BC} = \frac{DE}{EF}$.



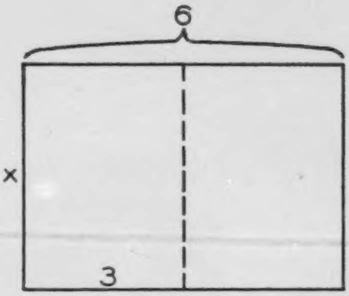
12. Three lots extend from Packard Street to State Street as shown in this drawing. The side lines make right angles with State Street, and the total frontage on Packard Street is 360'. Find the frontage of each lot on Packard Street.



13. Given: $\triangle$s ABC, XYZ, such that $\overleftrightarrow{XA}$, $\overleftrightarrow{YB}$, $\overleftrightarrow{ZC}$ meet in O and $\overleftrightarrow{AB} \parallel \overleftrightarrow{XY}$, $\overleftrightarrow{BC} \parallel \overleftrightarrow{YZ}$.
Prove: $\overleftrightarrow{AC} \parallel \overleftrightarrow{XZ}$.



14. A printer wishes to make a card 6 inches long and of such width that when folded on the dotted line as shown it will have the same shape as when unfolded. What should be the width?



Theorem 12-3. (The A.A.A. Similarity Theorem.) Given a correspondence between two triangles. If corresponding angles are congruent, then the correspondence is a similarity.

Restatement: Given a correspondence

$$ABC \longleftrightarrow DEF$$

between two triangles. If $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$, then

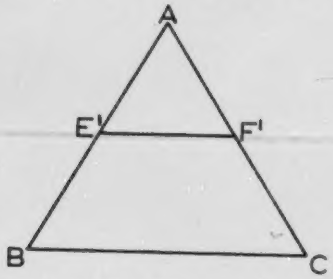
$$\triangle ABC \sim \triangle DEF.$$

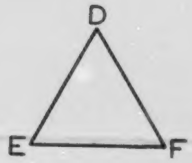
Notice that to prove that the correspondence is a similarity, we merely need to show that corresponding sides are proportional. (We don't need to worry about the angles, because corresponding angles are congruent by hypothesis). The proportionality of the sides means that

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

It will be sufficient to prove that the first of these equations always holds. (Exactly the same proof could then be repeated to show that the second equation also holds).

Thus we need to prove that $\frac{AB}{DE} = \frac{AC}{DF}$





Proof: Let E' and F' be points of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, such that $AE' = DE$ and $AF' = DF$. By the S.A.S. Postulate, we have

$$\triangle AE'F' \cong \triangle DEF.$$

Therefore $\angle AE'F' \cong \angle B$. Therefore $\overleftrightarrow{E'F'}$ and $\overleftrightarrow{BC}$ are parallel, or coincide. If they coincide then $\triangle AE'F' = \triangle ABC$, and so $\triangle ABC \cong \triangle DEF$; in this case,

$$AB = DE \text{ and } AC = DF,$$

or

$$\frac{AB}{DE} = \frac{AC}{DF} = 1.$$

If $\overleftrightarrow{E'F'}$ and $\overleftrightarrow{BC}$ are parallel, then by Theorem 12-1, we have

$$\frac{AB}{AE'} = \frac{AC}{AF'}.$$

But $AE' = DE$ and $AF' = DF$. Therefore

$$\frac{AB}{DE} = \frac{AC}{DF},$$

which was to be proved.

The theorem just proved allows us to prove a corollary which, it turns out, we quote oftener than the theorem in showing that two triangles are similar. Recall from Corollary 9-13-1 that if two pairs of corresponding angles of two triangles are congruent, the third pair must be also. Thus from Theorem 12-3 we immediately get the following corollary:

Corollary 12-3-1. (The A.A. Corollary.) Given a correspondence between two triangles. If two pairs of corresponding angles are congruent, then the correspondence is a similarity.

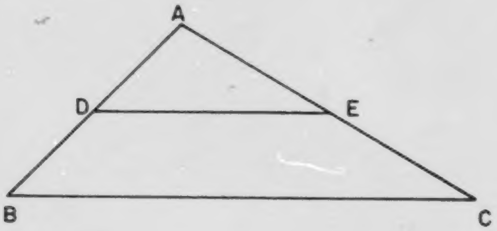
For example, if $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then

$$\triangle ABC \sim \triangle DEF.$$

If $\angle A \cong \angle D$ and $\angle C \cong \angle F$, then the same conclusion follows. And similarly for the third case.

We can now justify our statement at the beginning of this section by proving the following corollary:

Corollary 12-3-2. If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off a triangle similar to the given triangle.



For if $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ then by corresponding angles $\angle ADE \cong \angle B$ and $\angle AED \cong \angle C$. Also $\angle A \cong \angle A$. Hence $\triangle ADE \sim \triangle ABC$, by Theorem 12-3 or Corollary 12-3-1.

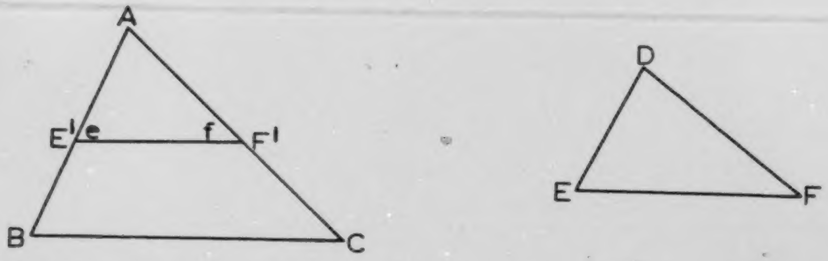
Theorem 12-4. (The S. A. S. Similarity Theorem.) Given a correspondence between two triangles. If two pairs of corresponding sides are proportional, and the included angles are congruent, then the correspondence is a similarity.

Restatement: Given $ABC \leftrightarrow DEF$.

If $$\angle A \cong \angle D$$

and $$\frac{AB}{DE} = \frac{AC}{DF},$$

then $$\triangle ABC \sim \triangle DEF.$$



Proof: Let E' and F' be points of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, such that $AE' = DE$ and $AF' = DF$. Then

$$\frac{AB}{AE'} = \frac{AC}{AF'}.$$

By Theorem 12-2, this means that $\overleftrightarrow{E'F'}$ and $\overleftrightarrow{BC}$ are parallel. When two parallel lines are cut by a transversal, corresponding angles are congruent. Therefore

$$\angle B \cong \angle e$$

and $$\angle C \cong \angle f.$$

But we know, by the S. A. S. Postulate, that

$$\triangle AE'F' \cong \triangle DEF.$$

Therefore $$\angle e \cong \angle E$$

and $$\angle f \cong \angle F.$$

Therefore $$\angle B \cong \angle E$$

and $$\angle C \cong \angle F.$$

We already knew by hypothesis that

$$\angle A \cong \angle D.$$

Therefore, by the A.A.A. Similarity Theorem, We have

$$\triangle ABC \sim \triangle DEF,$$

which was to be proved.

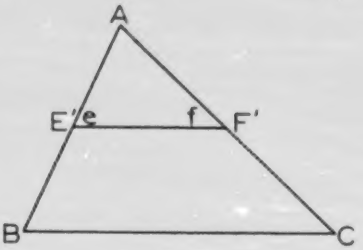
We have one more basic similarity theorem for triangles.

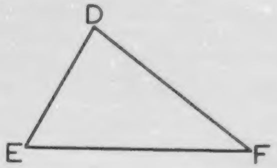
Theorem 12-5. (The S. S. S. Similarity Theorem.) Given a correspondence between two triangles. If corresponding sides are proportional, then the correspondence is a similarity.

Restatement: Given $ABC \leftrightarrow DEF$.

If $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$,

then $\Delta ABC \sim \Delta DEF$.





Proof: As before, let E' and F' be points of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, such that $AE' = DE$ and $AF' = DF$.

Statements	Reasons
1. $\frac{AB}{DE} = \frac{AC}{DF}$.	1. Hypothesis.
2. $\frac{AB}{AE'} = \frac{AC}{AF'}$.	2. Substitution.
3. $\overline{E'F'}$ and $\overline{BC}$ are parallel	3. Statement 2 and Theorem 12-2.
4. $\angle e \cong \angle B$ and $\angle f \cong \angle C$.	4. Theorem 9-9.
5. $\Delta ABC \sim \Delta AE'F'$.	5. A. A. Corollary.
6. $\frac{E'F'}{BC} = \frac{AE'}{AB}$.	6. Definition of similar triangles.
7. $E'F' = BC\frac{AE'}{AB} = BC\frac{DE}{AB}$.	7. Statement 6 and substitution.
8. $\frac{AB}{DE} = \frac{BC}{EF}$ or $EF = BC\frac{DE}{AB}$.	8. Hypothesis.

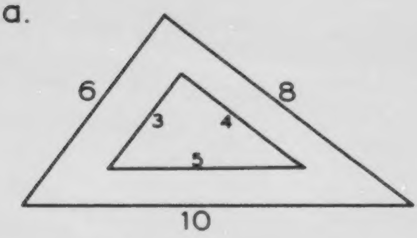
9.	$E'F' = EF$.	9.	Statements 7 and 8.
10.	$\Delta AE'F' \cong \Delta DEF$.	10.	The S.S.S. Theorem.
11.	$\angle e \cong \angle E$ and $\angle f \cong \angle F$.	11.	Corresponding parts.
12.	$\angle B \cong \angle E$ and $\angle C \cong \angle F$.	12.	Statements 4 and 11.
13.	$\Delta ABC \sim \Delta DEF$.	13.	The A.A. Corollary.

Problem Set 12-3b

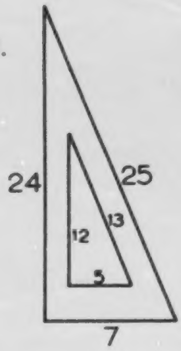
1. Given a correspondence $ABC \longleftrightarrow DEF$ between two triangles. Which of the following cases are sufficient to show that the correspondence is a similarity?
   a. $\angle A \cong \angle D$, $\angle B \cong \angle E$.
   b. $\frac{AB}{AC} = \frac{DE}{DF}$.
   c. Corresponding sides are proportional.
   d. Both triangles are equilateral.
   e. Both triangles are isosceles, and $m\angle A = m\angle D$.
   f. $m\angle C = m\angle F = 90$, and $AB = DE$.
2. Which of these similarity theorems do not have related congruence theorems: S.A.S., S.S.S., A.A.A., A.A.?
3. Is there any possibility of Δ I being similar to Δ II if:
   a. two angles of Δ I have measures of 60 and 70 while two angles of Δ II have measures of 50 and 80?
   b. two angles of Δ I have measures of 40 and 60 while two angles of Δ II have measures of 60 and 80?
   c. Δ I is a right Δ, while Δ II is isosceles with one angle of measure 40?
   d. Δ I has sides whose lengths are 5, 6, 7, while Δ II has a perimeter of 36,000.

4. Here are six pairs of triangles. In each case tell whether the two triangles are similar. If they are, state the theorem you would quote as proof.

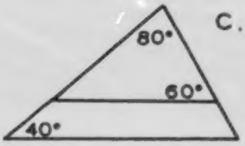
a.



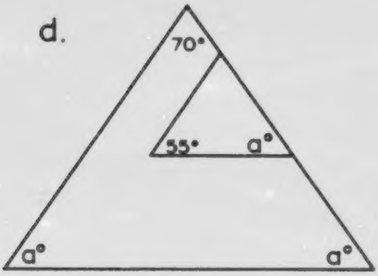
b.


24
25
12
13
5
7

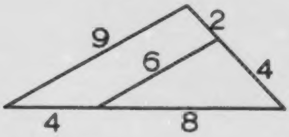
c.



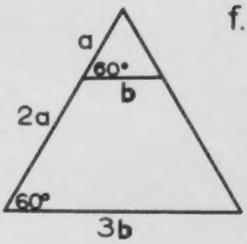
d.



e.



f.

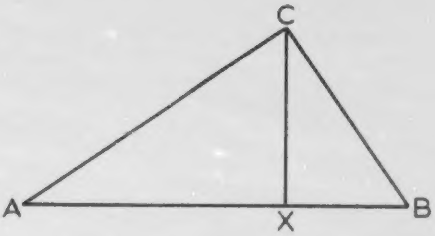

a
60°
b
2a
60°
3b

5. Given the figure shown with $\overline{AC} \perp \overline{BC}$ and $\overline{CX} \perp \overline{AB}$.

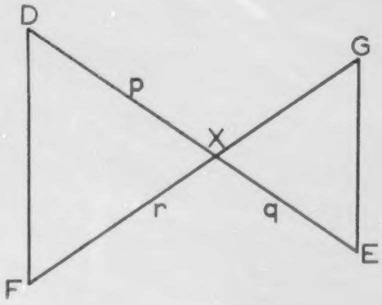
a. Name an angle which is congruent to ∠ ACB.

b. Name an angle with the same measure as ∠B.

c. Name a triangle which is similar to △ ACB.



6. If the lengths of $\overline{DX}$, $\overline{XE}$, and $\overline{FX}$ are p, q and r respectively, what length of $\overline{XG}$ will assure similarity of the triangles? If p = 3q, must m∠ D = 3m∠ E?



7. Below is a series of statements giving the lengths of sides of a number of triangles. Decide for each pair whether the triangles are similar and then make a statement as follows:

$\triangle$ _______ is similar to $\triangle$ _______, or

$\triangle$ _______ is not similar to $\triangle$ _______.

For each pair that are similar write a statement showing the proportionality of the sides.

a. AB = 5, AF = 3, FB = 7. QR = 15, QS = 9, RS = 21.

b. MT = 2, MW = 5, TW = 6. RS = $7\frac{1}{2}$, LS = 9, RL = 3.

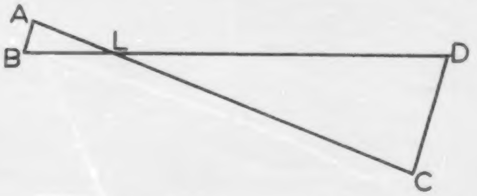
c. AB = 5, BC = 2, AC = 4. XY = $2\frac{1}{2}$, XZ = 2, YZ = 3.

d. AB = 6, AC = 7, BC = 8. RS = 40, RT = 35, ST = 30.

e. AB = 1.8, BC = 2.4, AC = 3.
XW = 0.4, XT = 0.5, WT = 0.3.

8. Given: $\angle B \cong \angle D$.
CD = 4AB.

Prove: BD = 5BL.



9.

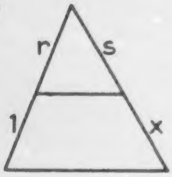


Fig. a.

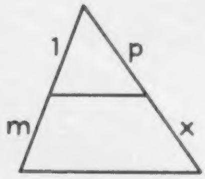


Fig. b.

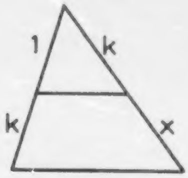


Fig. c

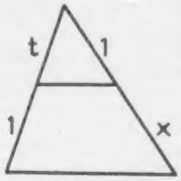


Fig. d

In each figure a segment has been drawn parallel to the base of a triangle, and the lengths of certain segments have been indicated.

a. Prove that $x = \frac{s}{r}$ (Hint: Write a proportion.)

b. Prove that $x = mp$.

c. Prove that $x = k^2$.

d. Prove: $x = \frac{1}{t}$.

e. Part c is a special case of which other part?

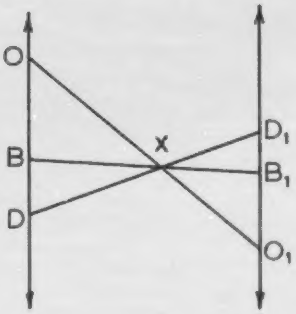
f. Part d is a special case of which other part?

g. Do the results depend on the size of the vertex angle?

10. Explain how two triangles can have five parts (sides, angles) of one triangle congruent to five parts of the other triangle, but not be congruent triangles.

11. Given: In the diagram $\overleftrightarrow{OD} \parallel \overleftrightarrow{O_1D_1}$.

Prove: $\frac{OB}{O_1B_1} = \frac{OD}{O_1D_1}$.

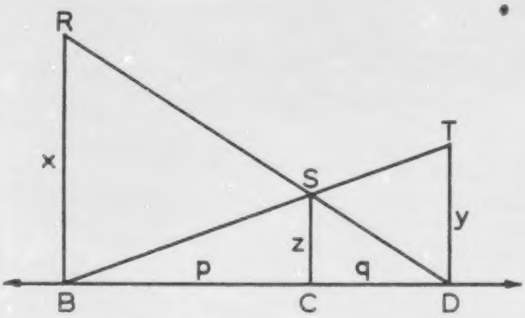


*12. a. If $\overline{BR}$, $\overline{CS}$ and $\overline{DT}$ are perpendicular to $\overline{BD}$, name the pairs of similar triangles.

b. Which is correct: $\frac{z}{y} = \frac{p}{q}$ or $\frac{z}{y} = \frac{p}{p+q}$?

c. Which is correct: $\frac{z}{x} = \frac{q}{p}$ or $\frac{z}{x} = \frac{q}{p+q}$?

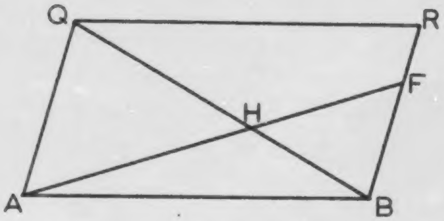
d. Show that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.



e. The problem, "How long does it take two men to complete a task which one alone can complete in 6 hours and the other alone in 3 hours?" can be answered by solving $\frac{1}{6} + \frac{1}{3} = \frac{1}{n}$. Solve this equation geometrically. (Hint: see part (d) and the figure.)

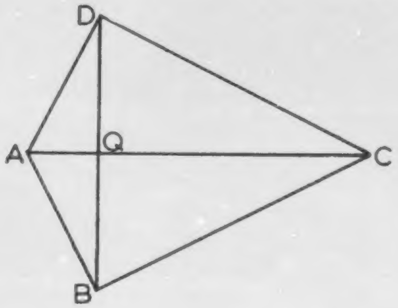
13. Given parallelogram ABRQ with diagonal $\overline{QB}$ and segment $\overline{AF}$ intersecting in H as shown.

Prove: $QH \cdot HF = HB \cdot AH$.



14. In this figure if $\overline{DB} \perp \overline{AC}$ and $DQ = BQ = 2AQ = \frac{1}{2} QC$.

Prove:
a. $\triangle AQD \sim \triangle DQC$.
b. $\triangle BQC \sim \triangle AQD$.
c. $\overline{AD} \perp \overline{DC}$.

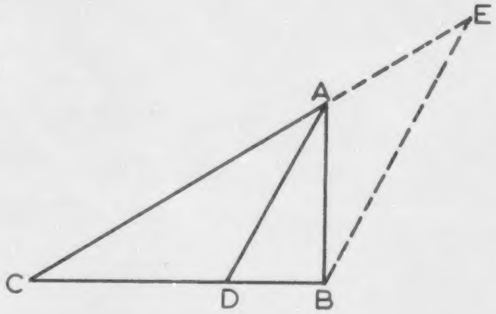


15. Prove the following theorem: the bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

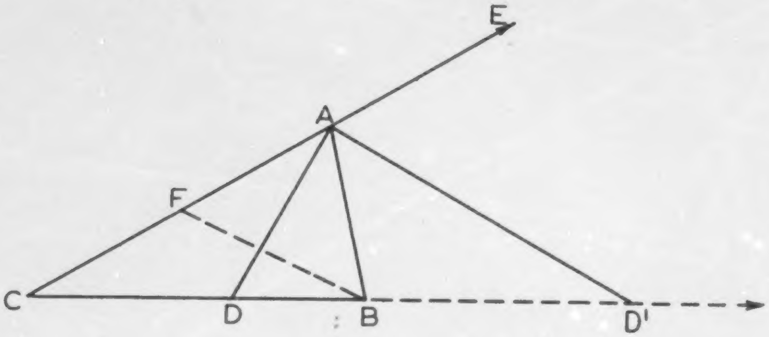
Given: $\triangle ABC$, $\overline{AD}$ the bisector of $\angle A$ meeting $\overline{BC}$ in D.

Prove: $\frac{CD}{DB} = \frac{CA}{AB}$.

(Hint: Make $\overline{BE} \parallel \overline{AD}$.)

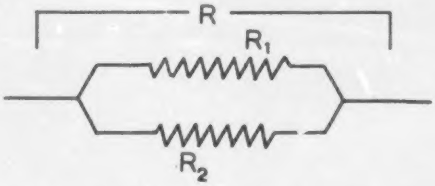


*16. Given ΔABC. Let the bisectors of the internal and external angles at A meet $\overleftrightarrow{CB}$ in points D and D' respectively. Prove that $\frac{CD'}{D'B} = \frac{CD}{DB}$. (Hint: Make $\overline{BF} \parallel \overline{D'A}$.)



*17. If we have an electrical circuit consisting of two wires in parallel, with resistances R_1 and R_2, then the resistance R of the circuit is given by the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$



The following scheme has been used to find R, given R_1 and R_2.

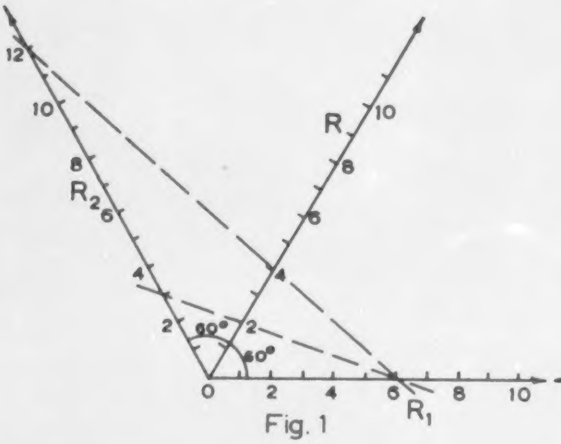

12
10
8
R₂
6
4
2
60°
2
60°
R
10
8
6
4
0
2
4
6
8
10
R₁

Fig. 1

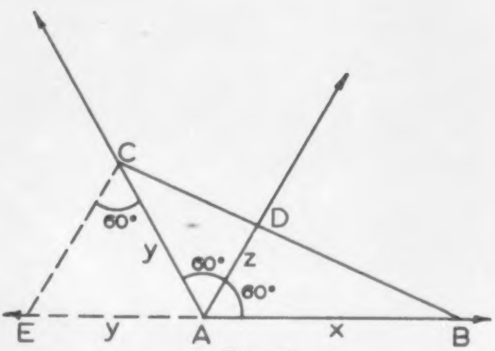

C
60°
y
60°
z
D
60°
E
y
A
x
B

Fig. 2

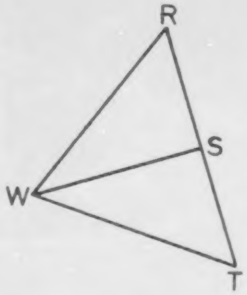
Numerical scales are marked off on three rays as in Figure 1. A straight-edge is placed so as to pass through R_1 and R_2 on the two outer scales, and R is read off on the third scale. Using the scales of the figure, select values for R_1, R_2, find R from the figure and check your result to see that the equation above is satisfied.

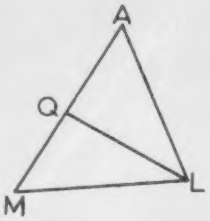
a. Prove that the method really works. See Figure 2.

b. Could the same diagram be used to find R in the equation $\frac{1}{R} = \frac{1}{R_1} - \frac{1}{R_2}$?

18. In this figure $\overline{WS}$ and $\overline{LQ}$ are medians and $\frac{RW}{AL} = \frac{RT}{AM} = \frac{WS}{LQ}$.

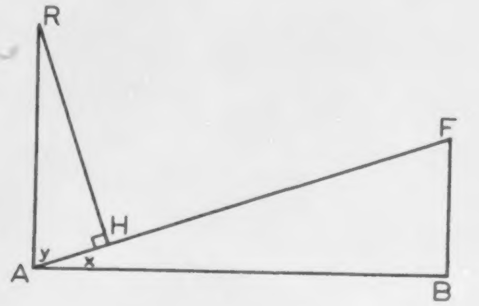
Prove that $\Delta RWT \sim \Delta ALM$.



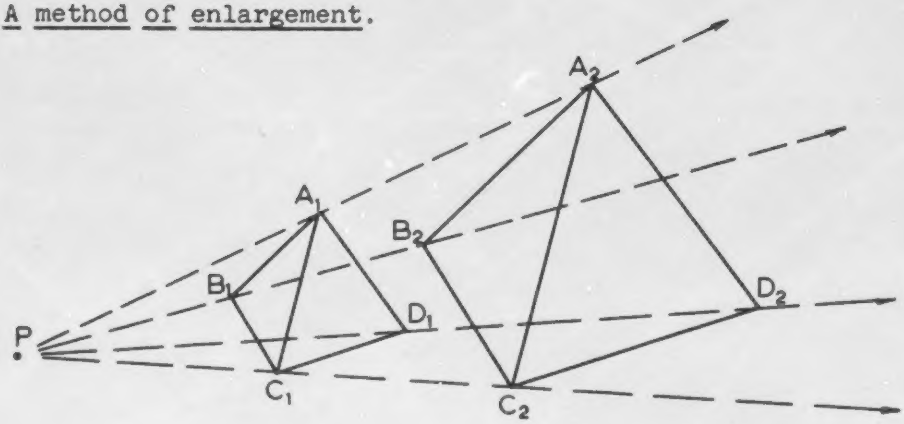


19. Given in this figure that $\overline{RA} \perp \overline{AB}$, $\overline{FB} \perp \overline{AB}$ and $\overline{RH} \perp \overline{AF}$.

Prove that $\Delta HRA \sim \Delta BAF$ and $HR \cdot BF = BA \cdot HA$.



20. A method of enlargement.



The figure $A_1B_1C_1D_1$ has been enlarged by introducing from an arbitrary point P, the rays $\overrightarrow{PA_1}$, $\overrightarrow{PB_1}$, $\overrightarrow{PC_1}$ and $\overrightarrow{PD_1}$; locating A_2, B_2, C_2 and D_2 so that $PA_2 = 2PA_1$, $PB_2 = 2PB_1$, etc.; and finally drawing segments $\overline{A_2B_2}$, $\overline{A_2D_2}$ etc.

a. Draw a simple object, a block or a table, for example, and enlarge it by the method shown. Is it necessary that PA_1, PB_1, etc. be doubled?

b. How could the method be modified to draw a figure with sides half the length of those of $A_1B_1C_1D_1$?

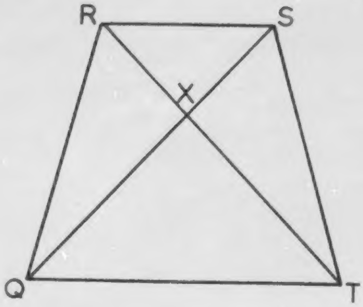
c. Prove: $\Delta PA_1B_1 \sim \Delta PA_2B_2$ and $\frac{A_2B_2}{A_1B_1} = \frac{PA_2}{PA_1}$.

d. Prove: $\Delta A_1B_1D_1 \sim \Delta A_2B_2D_2$.

e. Could the enlargement be carried out if P were selected on or inside the given figure?

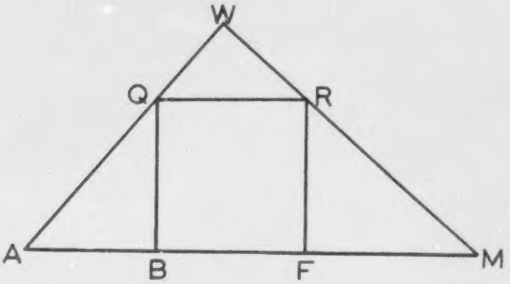
*21. Given: Quadrilateral RSTQ as in the figure with $\overline{RS} \parallel \overline{QT}$ and $\triangle QXR \sim \triangle TXS$.

Prove: $QR = TS$.

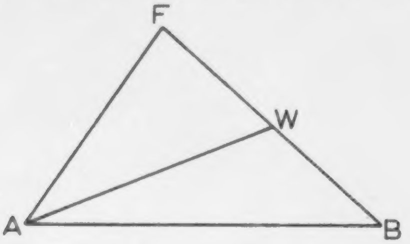


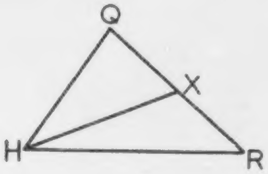
22. Given: $\overline{AW} \perp \overline{MW}$. BFRQ is a square with Q on $\overline{AW}$ and R on $\overline{WM}$ as shown in the figure.

Prove: $AB \cdot WR = QW \cdot BQ$,
and $AB \cdot FM = RF \cdot BQ$.

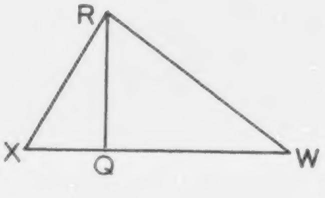


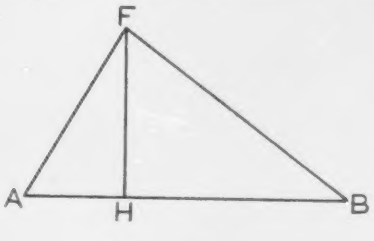
23. Prove the following theorem: In similar triangles corresponding medians have the same ratio as corresponding sides.





24. Prove the following theorem: In similar triangles corresponding altitudes have the same ratio as corresponding sides.

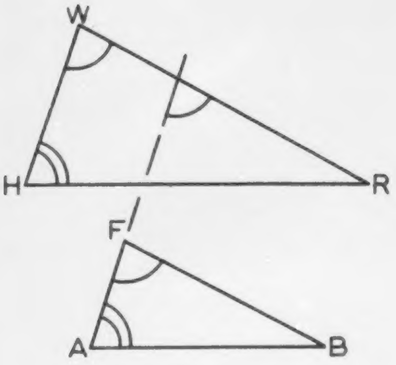




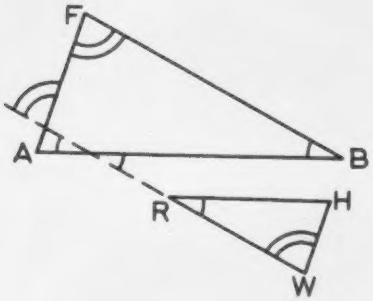
25. Prove that if the sides of two triangles are respectively parallel, the triangles are similar.

Given: $\overline{AB} \parallel \overline{HR}$.
$\overline{AF} \parallel \overline{HW}$.
$\overline{BF} \parallel \overline{RW}$..

Prove: $\triangle ABF \sim \triangle HRW$.

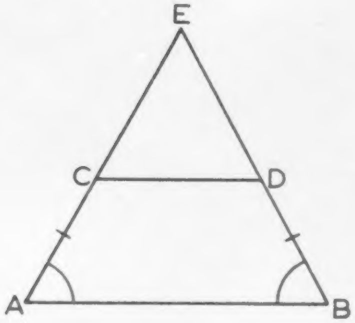


Case I

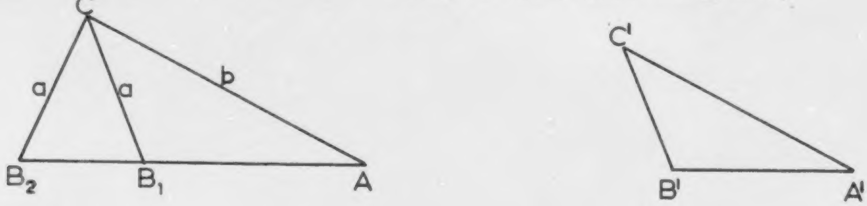


Case II

26. Given: $\angle A \cong \angle B$ and $AC = BD$.
Show $\overline{CD} \parallel \overline{AB}$



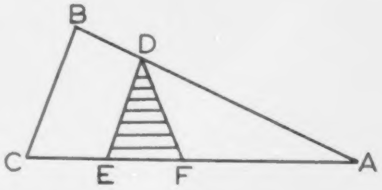
*27. It is known (see Chapter 5) that if two triangles correspond so that two sides and the angle opposite one of them in one triangle are congruent respectively to two sides and the angle opposite the corresponding side of the other (S. S. A.), the triangles need not be congruent. (See diagram.)



Is the following statement true or false? Explain.

If two triangles correspond such that two sides of one triangle are proportional to two sides of the other, and the angles opposite a pair of corresponding sides are congruent, then the triangles are similar.

*28. Δ EDF is isosceles with DE = DF. Δ ABC is such that E and F lie between A and C, $\overline{CB} \parallel \overline{ED}$, and A, B, D are collinear.



a. What true statements concerning similarity and proportions can be made concerning

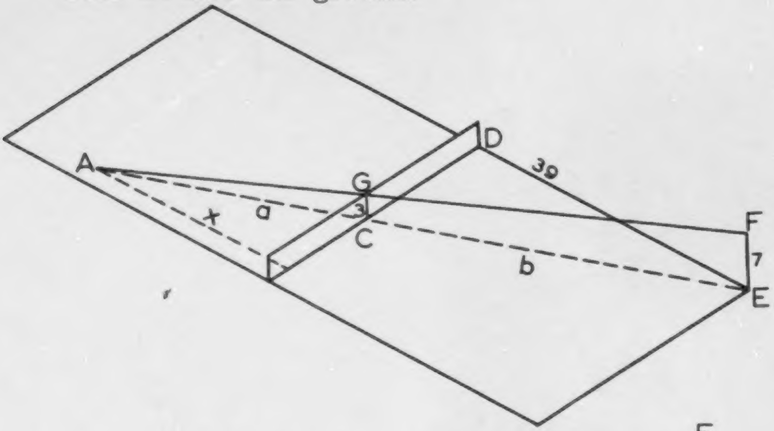
   1. Δ ABC and Δ ADE?

   2. Δ ABC and Δ ADF?

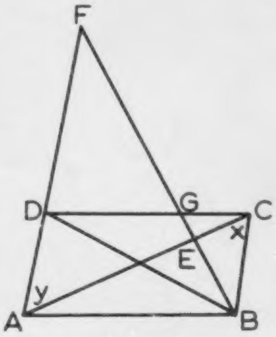
b. Is the following statement true or false? Explain.

   Given Δ ABC with D on segment $\overline{AB}$, X on segment $\overline{AC}$, such that $\frac{AB}{AD} = \frac{BC}{DX}$, then $\overleftrightarrow{BC}$ and $\overleftrightarrow{DX}$ must be parallel.

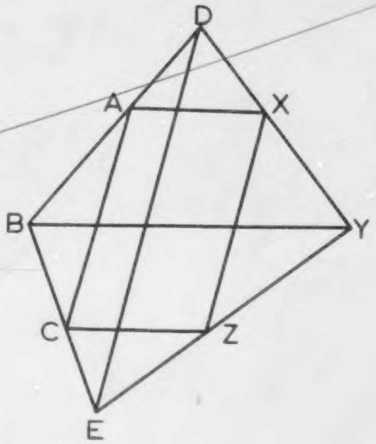
*29. A tennis ball is served from a height of 7 feet to clear a net 3 feet high. If it is served from a line 39 feet behind the net and travels in a straight path, how far from the net does it hit the ground?



*30. In the parallelogram ABCD shown in the figure the line $\overleftrightarrow{BF}$ intersects $\overleftrightarrow{AC}$ at E, $\overleftrightarrow{CD}$ at G, and $\overleftrightarrow{AD}$ at F. Prove that EB is the geometric mean of EG and EF.



*31. Given $\triangle$s ABC and XYZ such that $\overleftrightarrow{AX}$, $\overleftrightarrow{BY}$ and $\overleftrightarrow{CZ}$ are parallel and also $\overleftrightarrow{AC} \parallel \overleftrightarrow{XZ}$. $\overleftrightarrow{BA}$ and $\overleftrightarrow{YX}$ meet in D and $\overleftrightarrow{BC}$ and $\overleftrightarrow{YZ}$ meet in E.
Prove: $\overleftrightarrow{AC} \parallel \overleftrightarrow{DE} \parallel \overleftrightarrow{XZ}$.

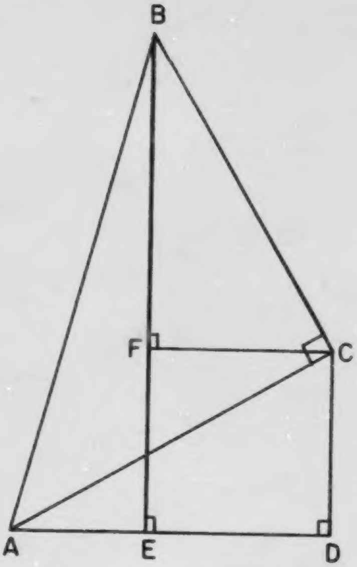


*32. The angles in the figure marked with small squares are right angles.

a. Show that $\frac{BF}{BC} = \frac{AD}{AC}$.

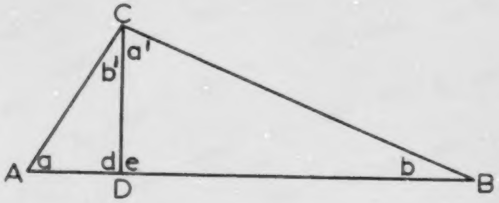
b. Then show that

$$\frac{BE}{AB} = \frac{CD}{AC} \cdot \frac{AC}{AB} + \frac{AD}{AC} \cdot \frac{BC}{AB}.$$



12-4. Similarities in Right Triangles.

Theorem 12-6. In any right triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar to each other and to the original triangle.



Restatement: Let Δ ABC be a right triangle with its right angle at C. Let $\overline{CD}$ be the altitude from C to the hypotenuse $\overline{AB}$. Then

$$\Delta ACD \sim \Delta ABC \sim \Delta CBD.$$

Notice that the restatement is more explicit than the first statement of the theorem; it tells us exactly how the vertices should be matched up to give the similarities. Notice also what the scheme is in matching up the angles: (1) The right angles match up with each other, as they have to in any similarity of right triangles. (2) Each little triangle has an angle in common with the big triangle, and so the angle matches itself. (3) The remaining angles are then matched.

Proof: In the proof, the notation for the angles will be as shown in the figure.

Since $\angle C$ is a right angle, we know that $\angle a$ and $\angle b$ are complementary. That is,

$$m\angle a + m\angle b = 90.$$

Also, since $\angle d$ is a right angle,

$$m\angle a + m\angle b' = 90.$$

Therefore $\angle b \cong \angle b'$.

Trivially, $\angle a \cong \angle a$;

and $\angle C \cong \angle d$,

because $\angle d$ is a right angle. By the A.A.A. Similarity Theorem, we have

$$\triangle ACD \sim \triangle ABC.$$

The proof of the other half of the theorem is precisely analogous, with the point B behaving like the point A.

Corollary 12-6-1. Given a right triangle and the altitude from the right angle to the hypotenuse:

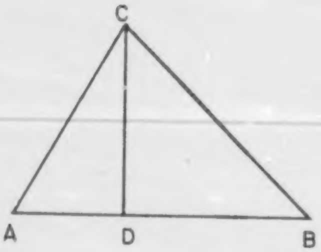
(1) The altitude is the geometric mean of the segments into which it separates the hypotenuse.

(2) Either leg is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

Restatement: Let $\triangle ABC$ be a right triangle with its right angle at C, and let D be the foot of the altitude from C to $\overline{AB}$. Then

(1) $\frac{AD}{CD} = \frac{CD}{BD}$.

(2) $\frac{AD}{AC} = \frac{AC}{AB}$ and $\frac{BD}{BC} = \frac{BC}{BA}$.



Proof: (1) By Theorem 12-6, $\triangle ADC \sim \triangle CDB$.

Hence, $\frac{AD}{CD} = \frac{CD}{BD}$.

(2) By Theorem 12-6, $\triangle ADC \sim \triangle ACB$.

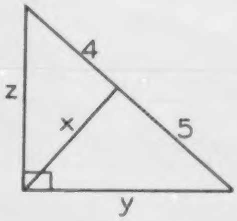
Hence, $\frac{AD}{AC} = \frac{AC}{AB}$.

Also, $\triangle BDC \sim \triangle BCA$,

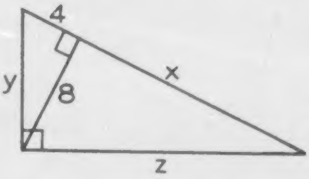
and so $\frac{BD}{BC} = \frac{BC}{BA}$.

Problem Set 12-4

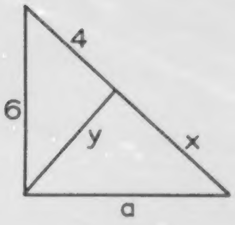
1. Given right $\triangle ABC$ with altitude drawn to the hypotenuse and lengths as shown, find the unknown lengths.



2. Follow the directions in Problem 1.



3. In this right triangle with the altitude drawn to the hypotenuse it is possible to find a numerical value for each segment a, x, y. Find them.



4. In a right triangle if the altitude to the hypotenuse is 12 and the hypotenuse is 25, find the length of each leg and of the segments of the hypotenuse.

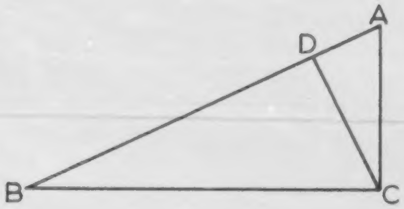
5. In right ΔABC, with right angle at C and altitude $\overline{CD}$,

   a. if $AD = 2$ and $DB = 8$, find AC, CD and CB.

   b. if $CD = 9$ and $AD = 3$, find AC, CB and AB.

   c. If $CB = 12$ and $AD = 10$, what are the lengths of the other segments?

   d. if $AC = 8$ and $DB = 12$, what are the lengths of the other segments?



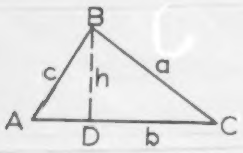
12-5. Areas of Similar Triangles.

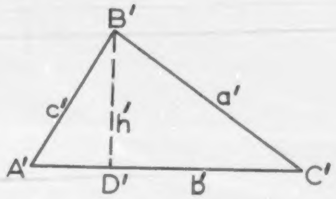
Given a square of side a, and a square of side $2a$, it is easy to see that the area of the second square is 4 times the area of the first. (This is because $(2a)^2 = 4a^2$.) In general, if two squares have sides a and ka, then the ratio of the areas is k^2, because

$$\frac{(ka)^2}{a^2} = \frac{k^2a^2}{a^2} = k^2.$$

An analogous result holds for similar triangles:

Theorem 12-7. The ratio of the areas of two similar triangles is the square of the ratio of any two corresponding sides.





Proof: Given $\Delta ABC \sim \Delta A'B'C'$. Then

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}.$$

Let k be the common value of these ratios, so that $a' = ka$, $b' = kb$, $c' = kc$. Let $\overline{BD}$ be the altitude from B to $\overleftrightarrow{AC}$, and let $\overline{B'D'}$ be the altitude from B' to $\overleftrightarrow{A'C'}$. Since ΔABD and $\Delta A'B'D'$ are right triangles, and $\angle A \cong \angle A'$, we have

$$\Delta ABD \sim \Delta A'B'D'.$$

Therefore

$$\frac{h'}{h} = \frac{c'}{c} = k.$$

Let A_1 and A_2 be the areas of the two triangles. Then

$$A_1 = \tfrac{1}{2}bh,$$

and

$$\begin{aligned} A_2 &= \tfrac{1}{2}b'h' \\ &= \tfrac{1}{2}(kb)(kh) \\ &= k^2 \cdot (\tfrac{1}{2}bh). \end{aligned}$$

Therefore $$\frac{A_2}{A_1} = k^2 = \left(\frac{a'}{a}\right)^2 = \left(\frac{b'}{b}\right)^2 = \left(\frac{c'}{c}\right)^2,$$

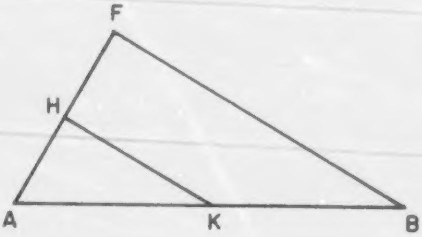
which was to be proved.

Problem Set 12-5

1. What is the ratio of the areas of two similar triangles whose bases are 3 inches and 4 inches? x inches and y inches?

2. A side of one of two similar triangles is 5 times the corresponding side of the other. If the area of the first is 6, what is the area of the second?

3. In the figure if H is the mid-point of $\overline{AF}$ and K is the mid-point of $\overline{AB}$, the area of ΔABF is how many times as great as the area of ΔAKH? If the area of ΔABF is 15, find the area of ΔAKH.



4. The area of the larger of two similar triangles is 9 times the area of the smaller. A side of the larger is how many times the corresponding side of the smaller?

5. The areas of two similar triangles are 225 sq. in. and 36 sq. in. Find the base of the smaller if the base of the larger is 20 inches.

6. The areas of two similar triangles are 144 and 81. If a side of the former is 6, what is the corresponding side of the latter?

7. In ΔABC, the point D is on side $\overline{AC}$, and AD is twice CD. Draw $\overline{DE}$ parallel to $\overline{AB}$ intersecting $\overline{BC}$ at E, and compare the areas of triangles ABC and DEC.

8. The edges of one cube are double those of another.

   a. What is the ratio of the sums of their edges?

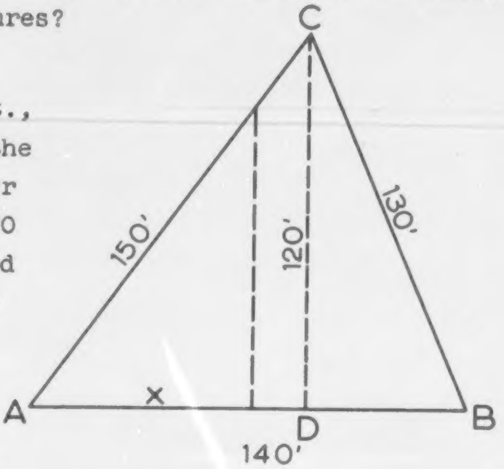
   b. What is the ratio of their total surface areas?

9. How long must a side of an equilateral triangle be in order that its area shall be twice that of an equilateral triangle whose side is 10?

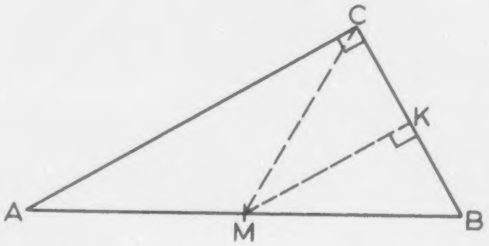
10. If similar triangles are drawn on the side and on the altitude of an equilateral triangle, so that the side and altitude are corresponding sides of the triangles, prove that their areas are to each other as 4 is to 3.

11. Two pieces of wire of equal length are bent to form a square and an equilateral triangle respectively. What is the ratio of the areas of the two figures?

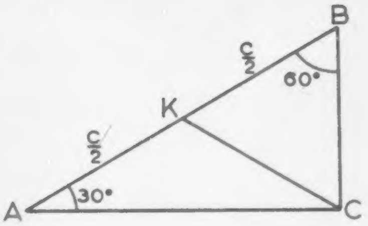
12. A triangular lot has sides with lengths 130 ft., 140 ft., and 150 ft. The length of the perpendicular from one corner to the side of 140 ft. is 120 ft. A fence is to be erected perpendicular to the side of 140 ft. so that the area of the lot is equally divided. How far from A along $\overline{AB}$ should this perpendicular be drawn?



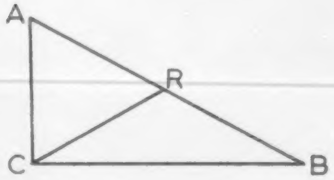
13. Prove the theorem: The midpoint of the hypotenuse of a right triangle is equidistant from the vertices.



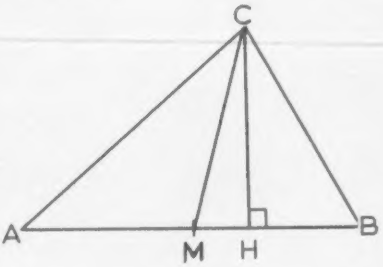
14. Prove Theorem 11-9 by using the following diagram and problem 13.



15. In this triangle AR = RC = RB. Prove that ΔABC is a right triangle.

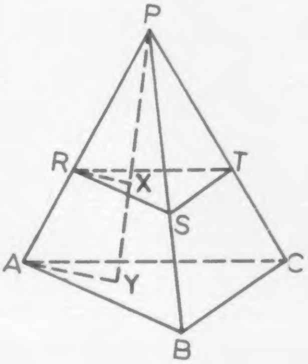


*16. Prove: The geometric mean of two positive numbers is less than their arithmetic mean, except when the two numbers are equal, in which case the geometric mean equals the arithmetic mean. (Hint: Let the two given numbers be the distances AH and HB, let $\overline{HC}$ be perpendicular to $\overline{AB}$, with $HC = \sqrt{AH \cdot HB}$, and let M be the mid-point of $\overline{AB}$. Prove $\angle ACB$ is a right angle and use the preceding two problems.)



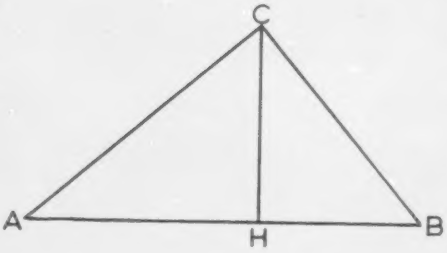
17. Given: P-ABC is a triangular pyramid with a section RST parallel to the base ABC. $\overline{PY}$ is perpendicular to the plane of the base, and X is the intersection of $\overline{PY}$ with the plane of Δ RST.

Prove: $\frac{\text{area } \Delta RST}{\text{area } \Delta ABC} = \left(\frac{PX}{PY}\right)^2$.



*18. In the figure, ΔABC is a right triangle, with hypotenuse $\overline{AB}$, and $\overline{CH}$ is the altitude from C.

Let the areas of ΔABC, ΔACH, ΔCBH be K_1, K_2, K_3 respectively.

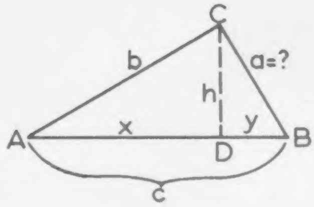


The following sequence of statements constitutes a different proof of the Pythagorean Theorem. Give a reason for each of the following statements:

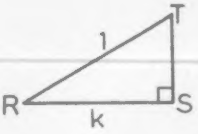
1. $K_1 = K_2 + K_3$.
2. $1 = \frac{K_2}{K_1} + \frac{K_3}{K_1}$.
3. $\Delta ACH \sim \Delta ABC \sim \Delta CBH$.
4. $1 = \left(\frac{AC}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2$.
5. $(AB)^2 = (AC)^2 + (BC)^2$.

Preamble. In the following problems, the lengths of two sides and the included angle of a triangle are given, and it is required to find the length of the third side. By the S.A.S. congruence theorem, the third side is uniquely determined, so there should be a method of finding it numerically. Another way of giving the included angle is to give a representative right triangle in which the angle (or its supplement) is one of the acute angles. Actually, only the number $k = \frac{RS}{RT}$ is needed. For numerical work, this number, which depends on $\angle R$, has been tabulated, and if this table is readily available the computation of the length of the third side is quite straightforward. The number k is called the cosine of $\angle R$, abbreviated $k = \cos \angle R$, and the table is called a table of cosines. For this reason the formula for a^2 that we find is called the law of cosines. You will meet it again in trigonometry.

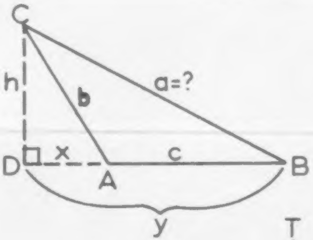
*19. In the two triangles shown in the diagram, $\angle A \cong \angle R$, $AC = b$, $AB = c$, $RS = k$ and $\angle S$ is a right angle. Find a in terms of b, c, and k.



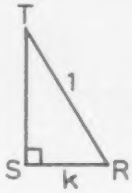
(Hint: Let D be the foot of the altitude to $\overline{AB}$, and let x, y, h be as indicated in the figure. Express a^2 in terms of h and y; express h and y in terms of x, b, and c; then, from the similarity $\Delta ADC \sim \Delta RST$, express x in terms of b and k.)



*20. In the two triangles shown in the diagram, $\angle BAC$ is the supplement of $\angle R$, and $AC = b$, $AB = c$, $RS = k$ and $\angle S$ is a right angle. Find a in terms of b, c and k.



(Hint: Let D be the foot of the perpendicular to $\overleftrightarrow{AB}$ from C. Then $\Delta ADC \sim \Delta RST$.)



*21. a. Let m_a be the length of the median to the side $\overline{BC}$ of ΔABC, and let $BC = a$, $CA = b$, $AB = c$. Prove that

$$m_a^2 = \frac{1}{2}\left(b^2 + c^2 - \frac{a^2}{2}\right).$$

b. Let m_a, m_b, m_c be the lengths of the medians of ΔABC, with sides of length a, b, c. Prove that

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

Review Problems

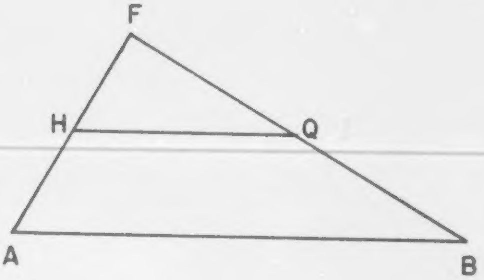
1. In the figure $\overline{HQ} \parallel \overline{AB}$.

   a. If FA = 11, FQ = 4, FH = 2, FB = ?

   b. If FB = 6, FH = 1, HA = 4, FQ = ?

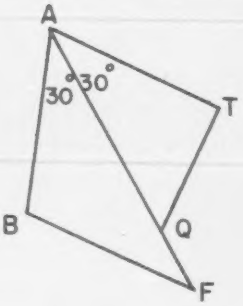
   c. If FA = 9, FB = 7, FH = $2\frac{1}{2}$, FQ = ?

   d. If HA = 6, FB = 12, FH = 3, QB = ?



2. a. Are the two triangles pictured here, similar if AB = 4, AF = 9, QF = 3, and AT = $2\frac{2}{3}$?

   b. If AB = 5, AT = 3, AQ = $4\frac{4}{5}$, what must AF be to make $\triangle TAQ \sim \triangle BAF$?

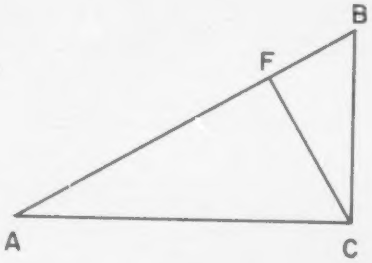


3. Give the geometric mean and the arithmetic mean for each of the following:

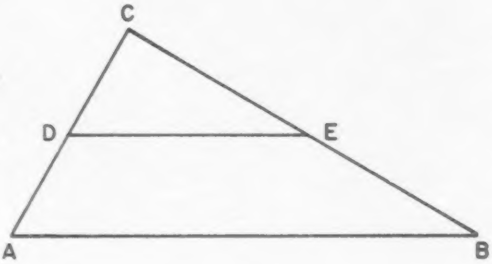
   a. 8 and 10 b. $6\sqrt{2}$ and $3\sqrt{2}$.

4. Sketch two figures which are not similar, but which have the sides of one proportional to the corresponding sides of the other.

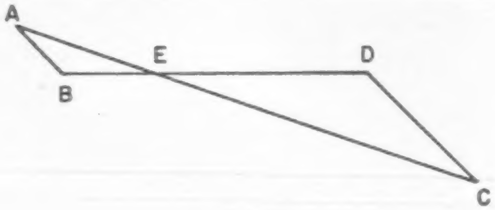
5. In right $\triangle$ ABC, if $\overline{FC}$ is the altitude to the hypotenuse, AF = 12 and BF = 3, find AC, FC and BC.



6. If $CD = x + 3$, $DA = 3x + 3$, $CE = 5$ and $EB = x + 5$, what must be the value of x to assure that $\overline{DE} \parallel \overline{AB}$?

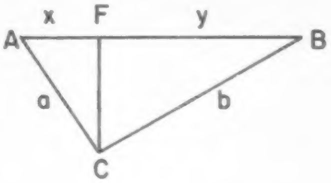


7. Given in this figure, $\angle B \cong \angle D$, $CD = 4AB$. Prove $BD = 5BE$.



8. A side of one equilateral triangle is congruent to an altitude of another equilateral triangle. What is the ratio of their areas?

9. In ΔABC, $\overline{AC} \perp \overline{BC}$, $\overline{CF} \perp \overline{AB}$, $AB = 20$ and $FC = 8$. Find a, b, x, and y.



10. If $\Delta ABC \sim \Delta DEF$ and $\Delta ACB \sim \Delta DEF$, show that $AB = AC$.

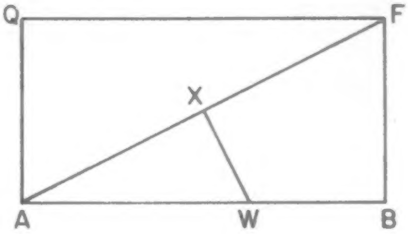
11. Given rectangle ABFQ as shown in the figure with $\overline{WX} \perp \overline{AF}$.

Prove:

a. $AF \cdot XW = AW \cdot QA$.

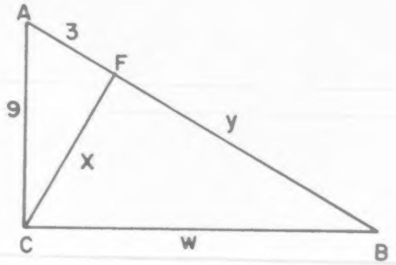
b. $QF \cdot XW = AX \cdot QA$.

c. $AF \cdot AX = AW \cdot QF$.



12. The tallest trees in the world are the redwoods along the coast of northern California. To measure one of these giants you move some distance from the tree and drive a stake in the ground. Then you hold a small mirror at ground level and sight it in, moving away from the stake until the top of the stake and the top of the tree are in a direct line.
If your stake is 5 feet tall and is 520 feet from the base of the tree, and if your mirror is 8 feet from the stake when the top of the stake and the top of the tree are in a straight line, how tall is the tree?

13. In right ΔABC with $\overline{CF}$ the altitude to the hypotenuse, and lengths as indicated in the figure, find x, y, and w.



*14. Join the vertices of ΔABC to a point R outside the triangle. Through any point X of $\overline{AR}$ draw $\overline{XY} \parallel \overline{AB}$ meeting $\overline{BR}$ at Y. Draw $\overline{YZ} \parallel \overline{BC}$ meeting $\overline{RC}$ at Z. Prove $\Delta XYZ \sim \Delta ABC$.

15. When we photograph a triangle, is the picture always similar to the original triangle? When can we be sure that it is?

Chapters 7 to 12

REVIEW EXERCISES

Write (1) if the statement is true and (0) if it is false. Be able to explain why you mark a statement false.

1. An exterior angle of a triangle is larger than any interior angle of the triangle.
2. In space there is only one perpendicular to a given line through a given external point.
3. The angle opposite the longest side of a triangle is always the largest angle.
4. In $\triangle ABC$, if $m\angle A < m\angle B$, then $AC < BC$.
5. If $\overline{AB} \perp \overline{BC}$, then $AB < AC$.
6. A triangle can be formed with sides of lengths 351, 513, and 135.
7. If an angle of one triangle is larger than an angle of a second triangle, then the side opposite the angle in the first is longer than the side opposite the angle in the second.
8. Two lines in space are parallel if they are both perpendicular to the same line.
9. Through every point in a plane there is a line parallel to a given line in the plane.
10. Given two lines and a transversal of them, if one pair of alternate interior angles are congruent, the other pair are also congruent.
11. If two lines are cut by a transversal so that one of two alternate interior angles is 90° larger than the other, the two lines are perpendicular.
12. If two lines are cut by a transversal, there are exactly four pairs of corresponding angles.

13. If two intersecting lines are cut by a transversal, no pair of corresponding angles are congruent.

14. If the alternate interior angles formed by two lines and a transversal are not congruent, the two lines are perpendicular.

15. Given two parallel lines and a transversal, two interior angles on the same side of the transversal are complementary.

16. If L, M and N are three lines such that $L \parallel M$ and $M \parallel N$, then $L \parallel N$.

17. If L, M and N are three lines such that $L \perp M$ and $M \perp N$, then $L \perp N$.

18. Since the sum of the measures of the angles of any triangle is 3 times 60, the sum of the measures of the angles of any quadrilateral is 4 times 60.

19. If two angles of one triangle are congruent to two angles of another triangle, then the third angles are congruent.

20. If two angles and a side of one triangle are congruent to two angles and a side of another, the triangles are congruent.

21. The acute angles in a right triangle are complementary.

22. An exterior angle of a triangle is the supplement of one of the interior angles of the triangle.

23. If a diagonal of a quadrilateral separates it into two congruent triangles, the quadrilateral is a parallelogram.

24. If each two opposite sides of a quadrilateral are congruent the quadrilateral is a parallelogram.

25. Opposite angles of a parallelogram are congruent.

26. A diagonal of a parallelogram bisects two of its angles.

27. A quadrilateral with three right angles is a rectangle.

28. The perimeter of the triangle formed by joining the midpoints of the sides of a given triangle is half the perimeter of the given triangle.

29. If the diagonals of a quadrilateral are perpendicular and congruent, the quadrilateral is a rhombus.

30. A set of parallel lines intercepts congruent segments on any transversal.

31. The area of a right triangle is the product of the hypotenuse and the altitude to the hypotenuse.

32. The area of a parallelogram is the product of the lengths of two of its adjacent sides.

33. The area of a trapezoid is half the product of its altitude and the sum of its bases.

34. If two triangles have equal area and equal bases, then they have equal altitudes.

35. If the legs of a right triangle have lengths a and b and the hypotenuse is of length c, then $b^2 = (c - a)(c + a)$.

36. If the lengths of the sides of a triangle are 20, 21 and 31, it is a right triangle.

37. Two right triangles are congruent if the hypotenuse and a leg of one are congruent respectively to the hypotenuse and a leg of the other.

38. If one of the angles of a right triangle contains 30°, then one leg is twice as long as the other leg.

39. The length of the diagonal of a square can be found by multiplying the length of a side by $\sqrt{2}$.

40. If a line intersecting two sides of a triangle cuts off a triangle similar to the larger one, the line is parallel to the third side of the triangle.

41. If each of two triangles have angles of 36° and 37°, the two triangles are similar.

42. If two triangles have an angle of one congruent to an angle of the other, and two sides of one proportional to two sides of the other, the triangles are similar.

43. If the sides of one triangle have lengths 6, 12, and 10, and the sides of another have lengths 15, 9 and 18, then the triangles are similar.

44. Any altitude of a right triangle separates it into similar triangles.

45. A triangle whose sides measure 4, 6 and 8 will have an area more than half the area of a triangle whose sides measure 6, 9 and 12.

46. If A, B, X, and Y are coplanar and if AX = BX and AY = BY, then $\overleftrightarrow{AB} \perp \overleftrightarrow{XY}$.

47. If three non-collinear points of a plane are each equidistant from points P and Q, then $\overleftrightarrow{PQ}$ is perpendicular to the plane.

48. If a line not contained in a plane is perpendicular to a line in a plane, then it is perpendicular to the plane.

49. A line perpendicular to each of two lines in a plane is perpendicular to the plane.

50. If a plane bisects a segment, every point of the plane is equidistant from the ends of the segment.

51. If a plane is perpendicular to each of two lines, the two lines are coplanar.

52. There are infinitely many planes perpendicular to a given line.

53. At a point on a line there are infinitely many lines perpendicular to the line.

54. Through a point outside a plane there is exactly one line perpendicular to the plane.

55. If a plane intersects two other planes in parallel lines, then the two planes are parallel.

56. Two planes perpendicular to the same line are parallel.

57. If plane E is perpendicular to $\overleftrightarrow{AB}$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, then $E \perp \overleftrightarrow{CD}$.

58. If each of two planes is parallel to a line, the planes are parallel to each other.

59. If a plane intersects the faces of a dihedral angle, the intersection is called a plane angle of the dihedral angle.

60. The projection of a line into a plane is always a line.

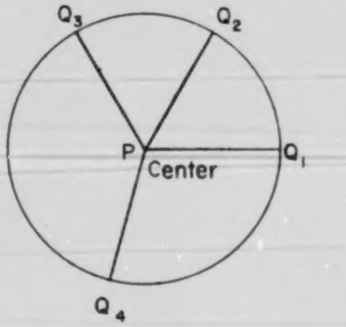
Chapter 13

CIRCLES AND SPHERES

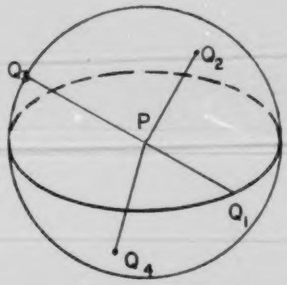
13-1. *Basic Definitions*.

In this chapter we commence the study of point sets not made up of planes, half-planes, lines, rays and segments. The simplest such *curved figures* are the circle and the sphere and portions of these. As usual in starting to talk about new figures we begin with some definitions.

Definitions: A *sphere* is the set of points each of which is at a given distance from a given point. A *circle* is the set of points in a given plane each of which is at a given distance from a given point of the plane. In each case the given point is called the *center* and the given distance *the radius* of the sphere or circle. Two or more spheres or circles with the same center are said to be *concentric*.



Circle



Sphere

$PQ_1 = PQ_2 = PQ_3 = PQ_4 =$ radius.

Theorem 13-1. The intersection of a sphere with a plane through its center is a circle with the same center and radius.

Proof: Since the sphere includes *all* points at a distance of the radius from the center, its intersection with a plane through the center will be the set of all points *in the plane* at this distance from the center; that is, the circle in this plane with the same center and radius.

Definition: The circle of intersection of a sphere with a plane through the center is called a *great circle* of the sphere.

There are two types of segments that are associated with spheres and circles.

Definitions: A *chord* of a circle or a sphere is a segment whose end-points are points of the circle or the sphere. The line containing a chord is a *secant*. A *diameter* is a chord containing the center. A *radius* is a segment one of whose end-points is the center and the other one a point of the circle or the sphere. The latter end-point is called the *outer end* of the radius.

The use of the word "radius" to mean both a segment and the length of that segment follows the convention introduced in Chapter 11. In the same way we use "diameter" to refer also to the length of a chord through the center as well as to the chord itself.

We may refer to a circle as circle C, or simply C. (C is most often used.) In stating problems it is convenient to use the convention that circle P denotes the circle with center P, provided there is no ambiguity as to which circle we mean. Similar remarks apply to spheres.

Problem Set 13-1

1. Study Section 13-1 to help you decide whether the following statements are true or false:
   - a. There is exactly one great circle of a sphere.
   - b. Every chord of a circle contains two points of the circle.
   - c. A radius of a circle is a chord of the circle.
   - d. The center of a circle bisects only one of the chords of the circle.
   - e. A secant of a circle may intersect the circle in only one point.
   - f. All radii of a sphere are congruent.
   - g. A chord of a sphere may be longer than a radius of the sphere.
   - h. If a sphere and a circle have the same center and intersect, the intersection is a circle.
2. Using your previous understanding of circles and spheres as well as your text, decide whether the following statements are true or false:
   - a. If a line intersects a circle in one point, it intersects the circle in two points.
   - b. The intersection of a line and a circle may be empty.
   - c. A line in the plane of a circle and passing through the center of the circle has two points in common with the circle.
   - d. A circle and a line may have three points in common.
   - e. If a plane intersects a sphere in at least two points, the intersection is a line.
   - f. A plane cannot intersect a sphere in one point.

g. If a plane intersects a radius of a sphere at its midpoint, the intersection of the plane and the sphere is a circle.

h. If two circles intersect, their intersection is two points.

3. A city is laid out in square blocks 100 yards on a side. Neglect the width of the streets in the following problems.

   a. Describe the location of the points which are 200 yards (as the crow flies) from a given street intersection.

   b. Describe the location of the points a taxi might reach by traveling 200 yards from a given street intersection. (City law prohibits U-turns.)

4. Prove the theorem: A diameter of a circle is its longest chord.

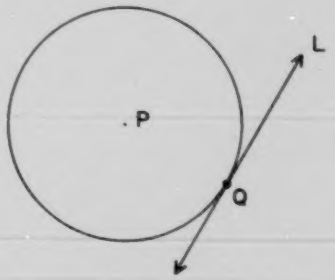
13-2. Tangent Lines. The Fundamental Theorem for Circles.

Definitions: The interior of a circle is the union of its center and the set of all points in the plane of the circle whose distances from the center are less than the radius. The exterior of the circle is the set of all points in the plane of the circle whose distances from the center are greater than the radius.

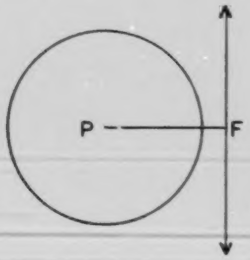
From these definitions it follows that a point in the plane of a circle is either in the interior of the circle, on the circle, or in the exterior of the circle. (We frequently use the more common word "inside" for "in the interior of", etc.)

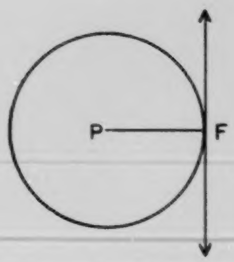
Definitions: A *tangent* to a circle is a line in the plane of the circle which intersects the circle in only one point. This point is called the *point of tangency*, or *point of contact*, and we say that the line and the circle *are tangent* at this point.

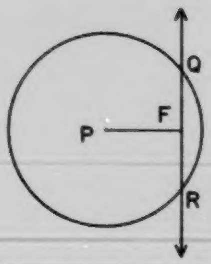
In the figure, L is tangent to the circle at Q.



We now want to find out what the possibilities are for a line and a circle in the same plane. It looks as if the following three figures ought to be a complete catalog of the possibilities:







In each case, P is the center of the circle, and F *is the foot of the perpendicular from* P *to the line*. We shall soon see that this point F -- the foot of the perpendicular -- is the key to the whole situation. If F is outside the circle, as in the first figure, then all other points of the line are also outside, and the line and the circle do not intersect at all. If F is on the circle, then the line is a tangent line, as in the second figure, and the point of tangency is F. If F is inside the circle, as in the third figure, then the line is a secant line, and the points of intersection are equidistant from the point F. To back all of this up, we need to prove the following theorem:

Theorem 13-2. Given a line and a circle in the same plane. Let P be the center of the circle, and let F be the foot of the perpendicular from P to the line. Then either

(1) Every point of the line is outside the circle, or

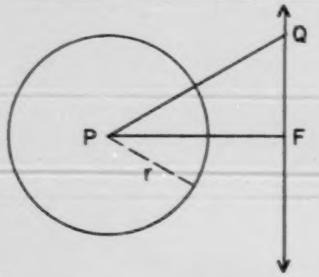
(2) F is on the circle, and the line is tangent to the circle at F, or

(3) F is inside the circle, and the line intersects the circle in exactly two points, which are equidistant from F.

This theorem is long, but its length is worthwhile, because once we have proved it, the hard part is over: all of the elementary theorems on secants, tangents and chords are corollaries of it.

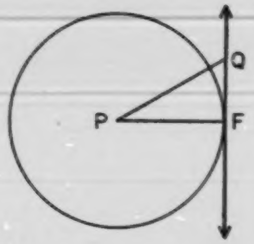
Proof: To prove the theorem, we shall show that if F is outside the circle, then (1) holds; if F is on the circle, then (2) holds; and if F is inside the circle, then (3) holds.

If F is outside the circle, then (1) holds.



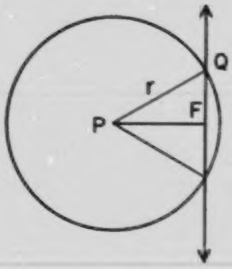
Let r be the radius of the circle. Then $PF > r$. By Theorem 7-6, the segment $\overline{PF}$ is the shortest segment joining P to the line. If Q is any other point of the line, then $PQ > PF$. Therefore, $PQ > r$, and Q is outside the circle.

If F is on the circle, then (2) holds.



Here we have PF = r. If Q is any other point of the line, then PQ > r. (Why?) Therefore the line is tangent to the circle, and the point of tangency is F.

If F is inside the circle, then (3) holds.



The proof is as follows. If Q is on both the line and the circle, then Δ PFQ is a right triangle with a right angle at F. By the Pythagorean Theorem,

$$PF^2 + FQ^2 = r^2,$$

so that

$$FQ^2 = r^2 - PF^2,$$

and

$$FQ = \sqrt{r^2 - PF^2}.$$

(The number under the radical is positive, because PF < r.) Thus any point Q common to the line and the circle must satisfy this last equation.

Conversely, any point Q lying on the line and satisfying this equation will be at distance r from P, as can be seen by going backwards through the algebra above. The equation

$$FQ = \sqrt{r^2 - PF^2}$$

is therefore the characterizing feature of the points Q which are intersections of the line and the circle.

By the Point Plotting Theorem there are exactly two such points, one on each of the two rays with end-point F. Obviously, they are equidistant from F.

This reasoning does not apply when the line passes through P, but in this case we have $P = F$, $PQ = FQ = r$, and there are two points Q as before.

Now we can proceed to our first basic theorems on tangents and chords which are all corollaries of Theorem 13-2. In all of these corollaries, it should be understood that C is a circle in a plane E, with center at P. To prove them, you merely need to refer to Theorem 13-2 and see which of the three conditions in the conclusion of the theorem applies to the case in hand.

Corollary 13-2-1. Every line tangent to C is perpendicular to the radius drawn to the point of contact.

Here it is Condition (2) that applies; and this means that the tangent and radius are perpendicular.

Corollary 13-2-2. Any line in E, perpendicular to a radius at its outer end, is tangent to the circle.

Since the outer end of the radius must be F, Condition (2) applies, and we have tangency.

Corollary 13-2-3. Any perpendicular from the center of C to a chord bisects the chord.

Here Condition (3) applies. (In Cases (1) and (2) there is no chord.)

Corollary 13-2-4. The segment joining the center of C to the mid-point of a chord is perpendicular to the chord.

Use Corollary 13-2-3 or Condition (3).

Corollary 13-2-5. In the plane of a circle, the perpendicular bisector of a chord passes through the center of the circle.

Use Corollaries 13-2-4 or 13-2-3, or Condition (3).

Corollary 13-2-6. If a line in the plane of a circle intersects the interior of the circle, then it intersects the circle in exactly two points.

Here also, Condition (3) applies. (In Case (1) and (2), the line doesn't intersect the interior of the circle.)

Definition: Circles of congruent radii are called *congruent*.

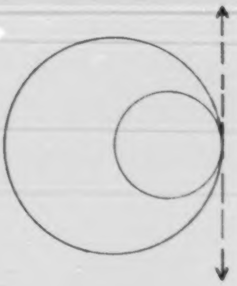
By the distance from a chord to the center of a circle we mean the distance between the center and the line containing the chord, as defined in Section 7-3. The proofs of the following two theorems are left to you:

Theorem 13-3. In the same circle or in congruent circles, chords equidistant from the center are congruent.

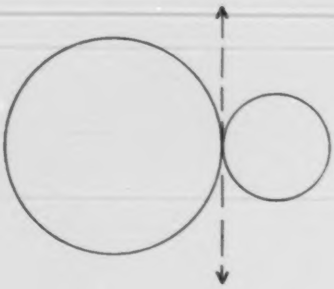
Theorem 13-4. In the same circle or in congruent circles, any two congruent chords are equidistant from the center.

The following additional definitions are useful in talking about circles and lines.

Definitions: Two circles are *tangent* if they are each tangent to the same line at the same point. If tangent circles are coplanar they are *internally* or *externally* tangent according as their centers lie on the same side or on opposite sides of the common tangent line.

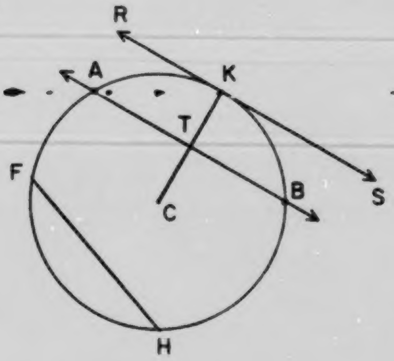


Internally tangent



Externally tangent

Problem Set 13-2



1. State the number of the theorem or corollary which justifies each conclusion below. (C is the center of the circle in the plane figure.)

   a. If TA = TB, then $\overline{CK} \perp \overline{AB}$.

   b. If $\overleftrightarrow{RS} \perp \overline{CK}$, then $\overleftrightarrow{RS}$ is tangent to the circle.

   c. If T is in the interior of the circle, then $\overleftrightarrow{KC}$ will intersect the circle in exactly one point other than point K.

   d. The perpendicular bisector of $\overline{FH}$ contains C.

   e. If $\overline{AB}$ and $\overline{FH}$ are equidistant from C, then $\overline{AB} \cong \overline{FH}$.

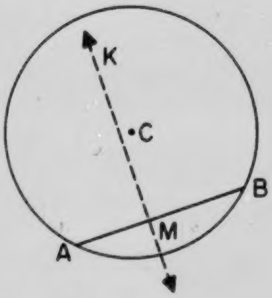
   f. If $\overleftrightarrow{RS}$ is tangent to circle C, then $\overline{CK} \perp \overleftrightarrow{RS}$.

   g. If $\overline{CK} \perp \overline{AB}$, then AT = TB.

   h. If $\overline{AB} \cong \overline{FH}$, then $\overline{AB}$ and $\overline{FH}$ are equidistant from C.

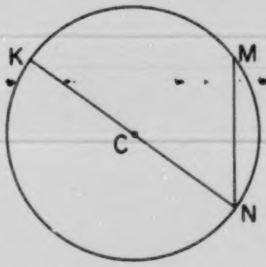
2. Prove Corollary 13-2-3: Any perpendicular from the center, C, of a circle to a chord bisects the chord.

3. Use this figure to prove Corollary 13-2-5: In the plane of a circle, the perpendicular bisector of a chord passes through the center of the circle.



4. Given a circle, how can its center be located?

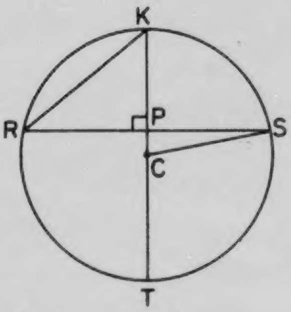
5. In circle C, KN = 40, and MN = 24. How far is $\overline{MN}$ from the center of the circle?



6. In a circle whose diameter is 30 inches a chord is drawn perpendicular to a radius. The distance from the intersection of chord and radius to the outer end of the radius is 3 inches. Find the length of the chord.

7. Given: The figure below, with C the center of the circle and $\overline{KT} \perp \overline{RS}$. In the ten problems respond as follows:
Write "A" if more numerical information is given than is needed to solve the problem.
Write "B" if there is insufficient information to solve the problem.
Write "C" if the information is sufficient and there is no unnecessary information.
Write "D" if the information given is contradictory.
(You do not need to solve the problems.)

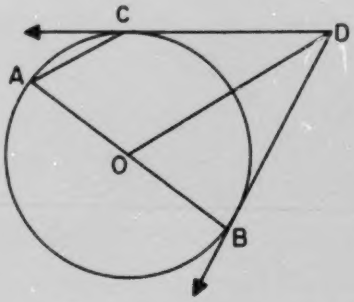
a. KP = 4, PC = 1, CT = 6, KT = ?
b. RP = 5, RS = ?
c. CT = 13, CP = 5, RS = ?
d. KP = 18, RS = 48, KC = 25, RK = ?
e. PC = 3.5, RS = 24, RK = ?
f. KT = 40, RP = 16, CS = ?
g. CS = 8, TK = 16, PC = ?
h. RK = 20, RS = 32, KP = 13, KT = ?
i. RS = 6, KC = 5, PT = ?
j. PT = 5, CS = 6, RS = ?



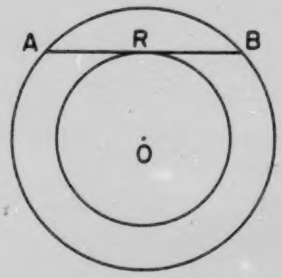
8. In a circle with center P a chord $\overline{AB}$ is parallel to a tangent and intersects the radius to the point of tangency at its mid-point. If AB = 12, find the radius of the circle.

9. Prove that the tangents to a circle at the ends of the diameter are parallel.

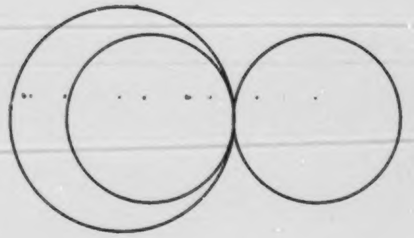
*10. In circle O with center at O, $\overline{AB}$ is a diameter and $\overline{AC}$ is any other chord from A. If $\overleftrightarrow{CD}$ is the tangent at C, and $\overleftrightarrow{DO} \parallel \overleftrightarrow{AC}$, prove that $\overleftrightarrow{DB}$ is tangent at B.



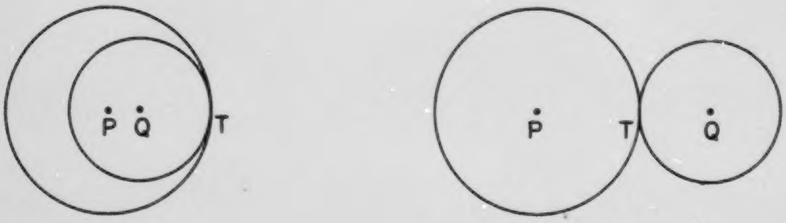
11. For the concentric circles of the figure, prove that all chords of the larger circle which are tangent to the smaller circle are bisected at the point of contact.
Restatement: In each circle the center is O. $\overline{AB}$, a chord of the larger circle, is tangent to the smaller circle at R.
Prove: AR = BR.



12. One arrangement of three circles so that any one is tangent to the other two is shown here. Make sketches to show three other arrangements of three circles with each circle tangent to the other two.



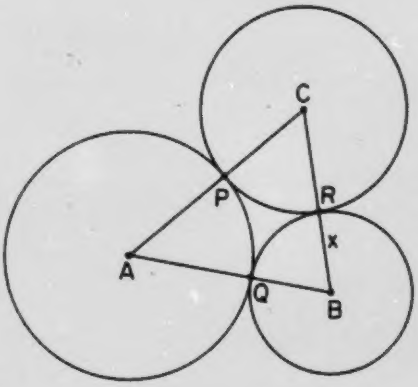
*13. Prove: The line of centers of two tangent circles contains the point of tangency. (Hint: Draw the common tangent.)



Case I

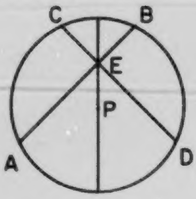
Case II

14. In the figure, A, B and C are the centers of the circles. AB = 14, BC = 10, AC = 18. Find the radius of each circle.

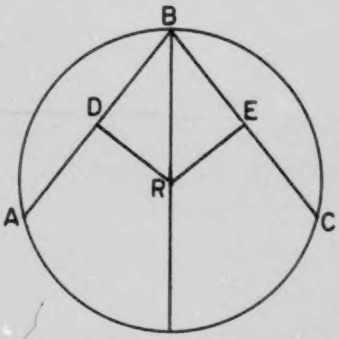


15. Prove Theorem 13-3: In the same circle or congruent circles, chords equidistant from the center are congruent.

*16. Given: In the figure P is the center of the circle, and $m\angle AEP = m\angle DEP$.
Prove: $\overline{AB} \cong \overline{CD}$.

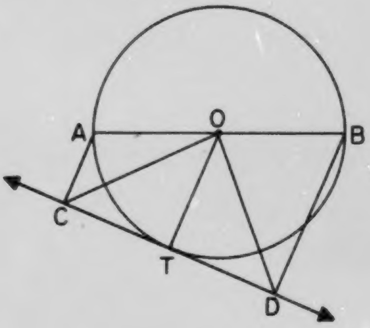


17. In circle R, $\overline{RD} \perp \overline{AB}$ and $\overline{RE} \perp \overline{BC}$, $RD = RE$.
Prove that $DA = EC$.



18. Prove: The mid-points of all congruent chords in any circle lie on a circle concentric with the original circle and with a radius equal to the distance of a chord from the center; and the chords are all tangent to this inner circle.

*19. Given: $\overline{AB}$ is a diameter of circle O. $\overleftrightarrow{CD}$ is tangent to O at T. $\overleftrightarrow{AC} \perp \overleftrightarrow{CD}$. $\overleftrightarrow{BD} \perp \overleftrightarrow{CD}$.
Prove: $\overline{CO} \cong \overline{DO}$.

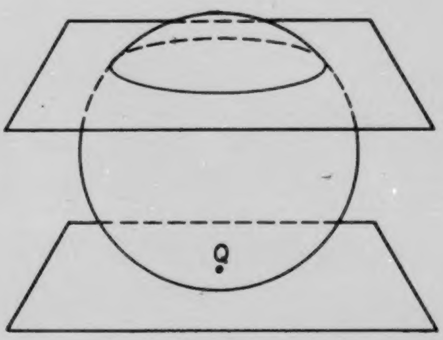


13-3. Tangent Planes. The Fundamental Theorem for Spheres.

Once you have studied and understood the last section, you should have very little trouble with this one. We shall see that spheres and planes in space behave in very much the same way as circles and lines in a plane, and the analogy between the theorems of the last section and the theorems of this section is very close indeed.

Definitions: The interior of a sphere is the union of its center and the set of all points whose distances from the center are less than the radius. The exterior of the sphere is the set of all points whose distances from the center are greater than the radius.

Definitions: A plane that intersects a sphere in exactly one point is called a tangent plane to the sphere. If the tangent plane intersects the sphere in the point Q then we say that the plane is tangent to the sphere at Q. Q is called the point of tangency, or the point of contact.



The basic theorem relating spheres and planes is the following:

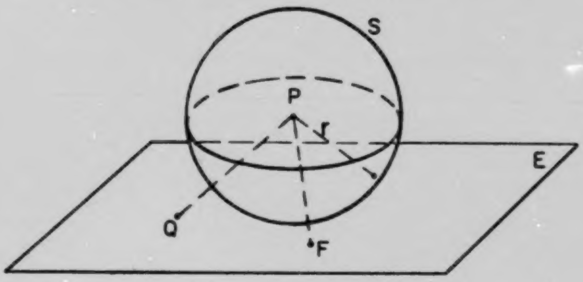
Theorem 13-5. Given a plane E and a sphere S with center P. Let F be the foot of the perpendicular segment from P to E. Then either

(1) Every point of E is outside S, or

(2) F is on S, and E is tangent to S at F, or

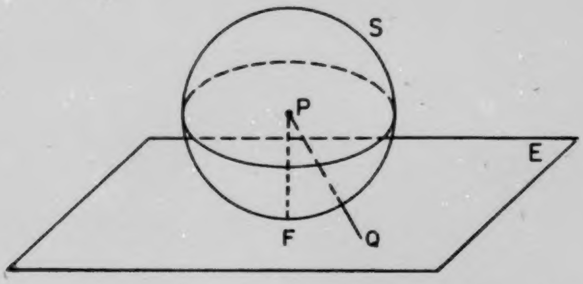
(3) F is inside S, and E intersects S in a circle with center F.

Proof: If F is outside S then (1) holds.



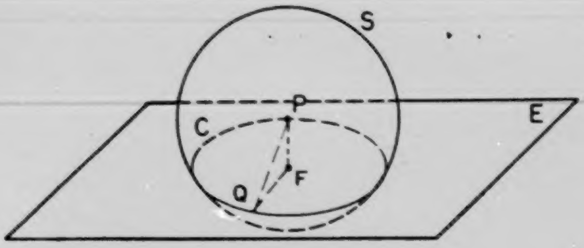
The proof follows almost word for word the corresponding proof for the circle in Theorem 13-2. The only significant change is the use of Theorem 8-11 (shortest segment from point to plane) instead of Theorem 7-6.

If F is on S then (2) holds.

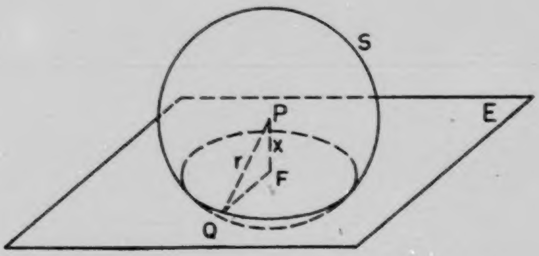


Here, again, the proof is almost identical with that of Theorem 13-2.

If F is inside S then (3) holds.



Let Q be any point which lies on both E and S. Let r be the radius of S, and let $x = PF$.



Then $\angle PFQ$ is a right angle, because every line in E, through F, is perpendicular to $\overleftrightarrow{PF}$. Therefore

$$FQ^2 + x^2 = r^2,$$

and

$$FQ = \sqrt{r^2 - x^2}.$$

Since Q is any point of the intersection of E and S, then every point Q of the intersection is such that FQ is constant. Therefore every point of the intersection lies on the circle with center at F and radius $\sqrt{r^2 - x^2}$.

Although we have shown that every point of the intersection is on the circle, we have not shown that this set of points is the circle. That is, there conceivably could be some points of the circle which are not points of the intersection. We now prove that this is not possible by showing that if Q lies on the circle, then it must be a point of the intersection.

Suppose that Q lies on the circle with center F and radius $\sqrt{r^2 - x^2}$. Then $\angle PFQ$ is a right angle, as before, so that

$$PQ^2 = x^2 + \left(\sqrt{r^2 - x^2}\right)^2 = r^2,$$

$$PQ = \sqrt{r^2} = r, \quad \text{since} \quad r > 0.$$

Therefore Q lies on the sphere. Therefore every point of the circle lies in the intersection. Therefore the circle is precisely the intersection, which was to be proved.

Our first basic theorems on tangents to a sphere are all corollaries of Theorem 13-5. In all of these corollaries, it should be understood that S is a sphere with center at P.

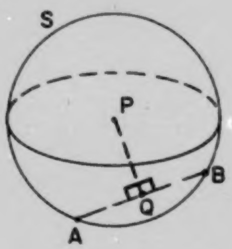
Corollary 13-5-1. A plane tangent to S is perpendicular to the radius drawn to the point of contact.

Corollary 13-5-2. A plane perpendicular to a radius at its outer end is tangent to S.

Corollary 13-5-3. A perpendicular from P to a chord of S bisects the chord.

Given: $\overline{PQ} \perp \overline{AB}$.

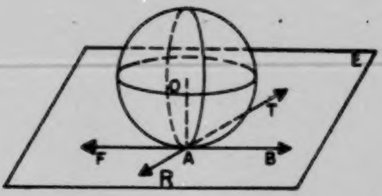
Prove: $AQ = BQ$.



Corollary 13-5-4. The segment joining the center of S to the mid-point of a chord is perpendicular to the chord.

Problem Set 13-3

1. Sphere O is tangent to plane E at A. $\overleftrightarrow{FB}$ and $\overleftrightarrow{RT}$ are lines of E through A. What is the relationship of $\overleftrightarrow{OA}$ to $\overleftrightarrow{FB}$ and $\overleftrightarrow{RT}$?

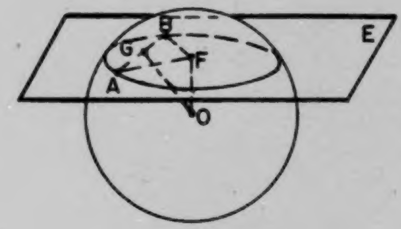


2. In a sphere having a radius of 10, a segment from the center perpendicular to a chord has length 6. How long is the chord?

3. In a sphere whose radius is 5 inches, what is the radius of a circle made by a plane 3 inches from the center?

4. Prove that circles formed on a sphere by planes equidistant from the center of the sphere are congruent.

*5. In the figure, plane E intersects the sphere having center O. A and B are two points of the intersection. F lies in plane E. $\overline{OF} \perp E$. $\overline{AF} \perp \overline{BF}$. If $AB = 5$ and $OF = AF$, find the radius of the sphere and $m\angle AOB$. If G is the mid-point of $\overline{AB}$, find OG.



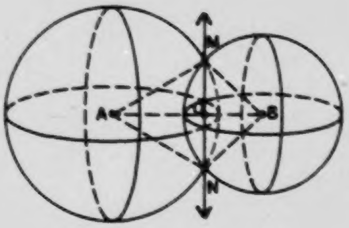
*6. Given a sphere and three points on it. Explain how to determine the center and the radius of the circle which the points determine. Explain how to determine the center and radius of the sphere.

*7. Given that plane E is tangent to a sphere S at point T. Plane F is any plane other than E which contains T. Prove (a) that plane F intersects sphere S and plane E in a circle and a line respectively; and (b) that the line of intersection is tangent to the circle of intersection.

8. Show that any two great circles of a sphere intersect at the end-points of a diameter of the sphere.

*9. Two great circles are said to be perpendicular if they lie in perpendicular planes. Show that, given any two great circles, there is one other great circle perpendicular to both. If two great circles on the earth are meridians (through the poles), what great circle is their common perpendicular?

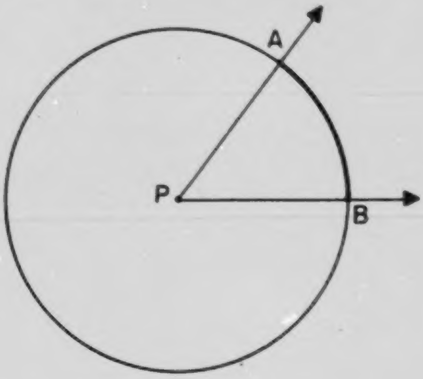
*10. In the figure, A and B are the centers of two intersecting spheres. Briefly describe the intersection. M and N are points of the intersection. O is a point in the plane of the intersection and is collinear with A and B. If the radius of sphere A is 13, the radius of sphere B is $5\sqrt{2}$, and $\overline{MB} \perp \overline{NB}$, find the distance between the centers of the spheres.



13-4. Arcs of Circles.

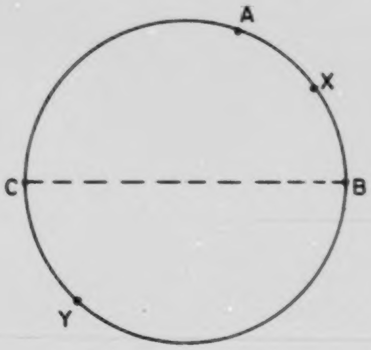
So far in this chapter we have been able to treat circles and spheres in similar manners. For the rest of this chapter we will confine ourselves exclusively to circles. The topics we will discuss have their corresponding analogies in the theory of spheres but these are too complicated to consider in a beginning course.

Definition: A central angle of a given circle is an angle whose vertex is the center of the circle.



Definitions: If A and B are two points of a circle with center P, not the end-points of a diameter, the union of A, B, and all the points of the circle in the interior of $\angle APB$ is a minor arc of the circle. The union of A, B, and all points of the circle in the exterior of $\angle APB$ is a major arc of the circle. If $\overline{AB}$ is a diameter the union of A, B, and all points of the circle in one of the two half-planes lying in the plane of the circle with edge $\overleftrightarrow{AB}$ is a semi-circle. An arc is either a minor arc, a major arc or a semi-circle. A and B are the end-points of the arc.

An arc with end-points A and B is most easily denoted by $\widehat{AB}$. This simple notation is always ambiguous, for even on the same circle there are always two arcs with A and B as end-points. Sometimes it will be plain from the context which arc is meant. If not, we will pick another point X somewhere in the arc $\widehat{AB}$, and denote the arc by $\widehat{AXB}$.

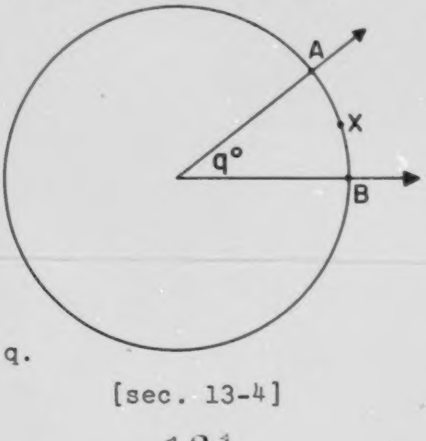


For example, in the figure, $\widehat{AXB}$ is a minor arc; $\widehat{AYB}$ is the corresponding major arc; and the arcs $\widehat{CAB}$ and $\widehat{CYB}$ are semi-circles.

The reason for the names "minor" and "major" is apparent when one draws several arcs of each kind. A major arc is, in an intuitive sense, "bigger" than a minor arc. This relation will be made more explicit in our next definition.

Definition: The degree measure $m\widehat{AXB}$ of an arc $\widehat{AXB}$ is defined in the following way:

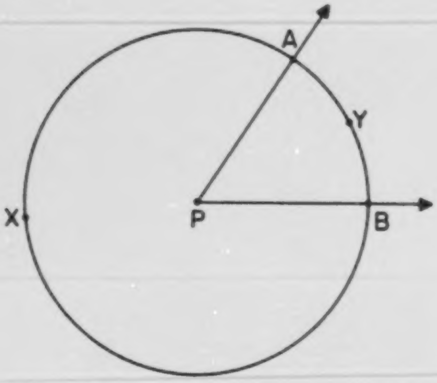
(1) If $\widehat{AXB}$ is a minor arc, then $m\widehat{AXB}$ is the measure of the corresponding central angle.



$m\widehat{AXB} = q$.

(2) If $\widehat{AXB}$ is a semi-circle, then $m\widehat{AXB} = 180$.

(3) If $\widehat{AXB}$ is a major arc, and $\widehat{AYB}$ is the corresponding minor arc, then $m\widehat{AXB} = 360 - m\widehat{AYB}$.

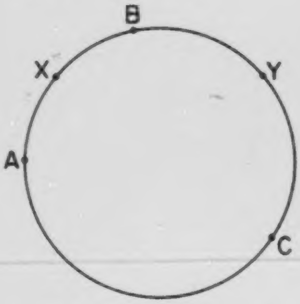


In the figure, $m\angle APB$ is approximately 60. Therefore $m\widehat{AYB}$ is approximately 60, and $m\widehat{AXB}$ is approximately 300.

Hereafter, $m\widehat{AXB}$ will be called simply the measure of the arc $\widehat{AXB}$. Note that an arc is minor or major according as its measure is less than or greater than 180.

The following theorem is simple and reasonable, but its proof is surprisingly tedious. We will state it without proof, and regard it, for practical purposes, as a postulate:

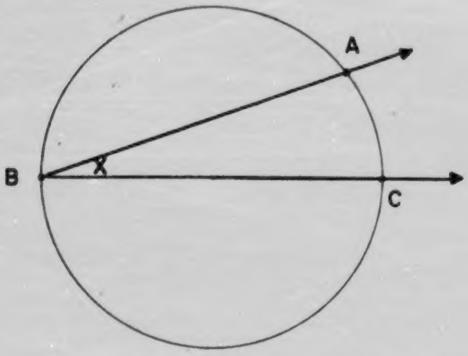
Theorem 13-6. If $\widehat{AB}$ and $\widehat{BC}$ are arcs of the same circle having only the point B in common, and if their union is an arc $\widehat{AC}$, then $m\widehat{AB} + m\widehat{BC} = m\widehat{AC}$.

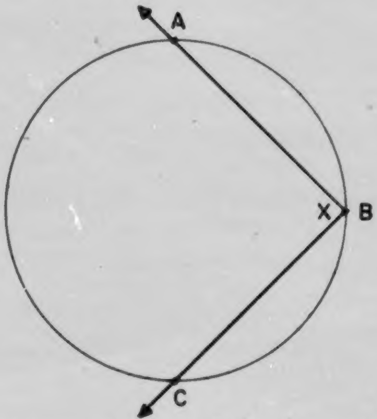


$m\widehat{AXB} + m\widehat{BYC} = m\widehat{ABC}$.

Notice that for the case in which $\widehat{AC}$ is a minor arc, the theorem follows from the Angle Addition Postulate. The proof in the general case is more troublesome.

In each of the figures below, the angle x is said to be inscribed in the arc $\widehat{ABC}$.

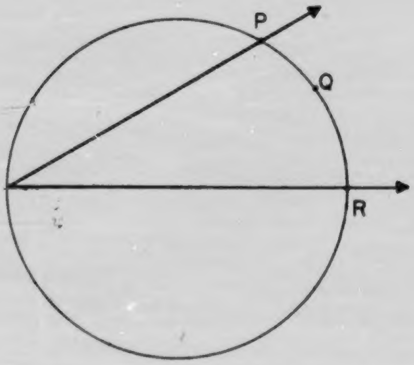


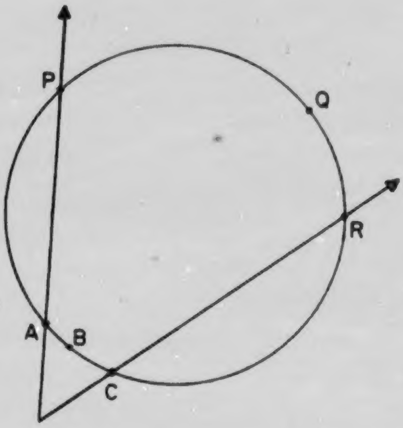


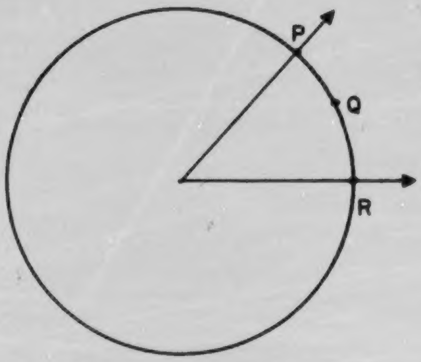
Definition: An angle is inscribed in an arc if (1) the two end-points of the arc lie on the two sides of the angle and (2) the vertex of the angle is a point, but not an end-point, of the arc. More concisely, $\angle ABC$ is inscribed in $\widehat{ABC}$.

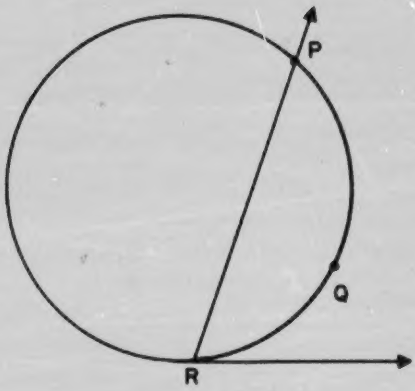
In the first figure, the angle is inscribed in a major arc, and in the second figure the angle is inscribed in a semi-circle.

In each of the figures below, the angle shown is said to intercept $\widehat{PQR}$.







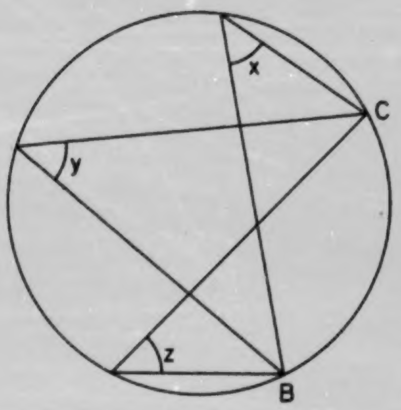


In the first case, the angle is inscribed; in the second case, the vertex is outside the circle; in the third case, the angle is a central angle; and in the last case, one side of the angle is tangent to the circle. In the second case, the angle shown intercepts not only the arc $\overparen{PQR}$ but also the arc $\overparen{ABC}$.

These figures give the general idea. We will now give the definition of what it means to say that an angle intercepts an arc. You should check very carefully to make sure that the definition really takes care of all four of the above cases.

Definition: An angle intercepts an arc if (1) the end-points of the arc lie on the angle, (2) each side of the angle contains at least one end-point of the arc and (3) except for its end-points, the arc lies in the interior of the angle.

The reason why we talk about the arcs intercepted by angles is that under certain conditions there is a simple relation between the measure of the angle and the measure of the arc.

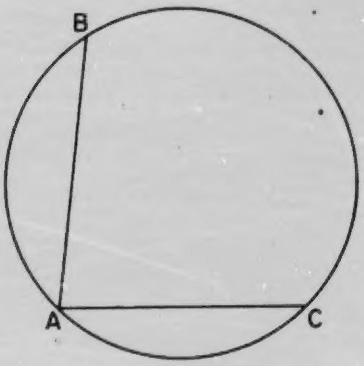


In the figure above we see three inscribed angles, ∠ x, ∠ y, ∠ z, all of which intercept the same arc $\widehat{BC}$. It looks as if these three angles are congruent. Indeed, it is a fact that this is what always happens. This fact is a corollary of the following theorem:

Theorem 13-7. The measure of an inscribed angle is half the measure of its intercepted arc.

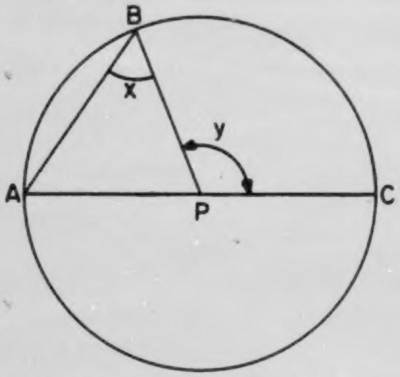
Restatement: Let ∠ A be inscribed in an arc of a circle, intercepting the arc $\widehat{BC}$. Then

$$m\angle A = \frac{1}{2} m\widehat{BC}.$$



In order to prove this from our previous theorems we first consider an angle inscribed in a special way.

Proof: Case 1. Suppose that one side of $\angle A$ contains a diameter of the circle, like this:



Let $\angle x$ and $\angle y$ be as in the figure. Then

$$m\angle A + m\angle x = m\angle y,$$

by Corollary 9-13-3. PA = PB, because A and B lie on the circle. Since the base angles of an isosceles triangle are congruent, we have $m\angle A = m\angle x$.

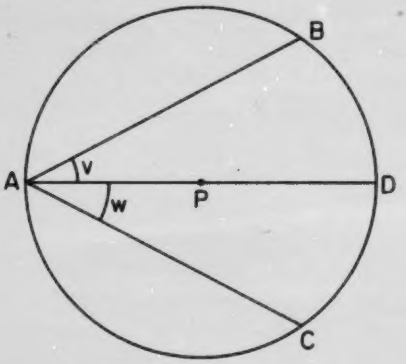
Therefore $$2(m\angle A) = m\angle y,$$

and $$m\angle A = \frac{1}{2}(m\angle y) = \frac{1}{2}(m\widehat{BC}),$$

which was to be proved.

Now we know that the theorem always holds in Case 1. Using this fact, we show that the theorem holds in every case.

Case 2. Suppose that B and C are on opposite sides of the diameter through A, like this:



Then $$m\angle A = m\angle v + m\angle w,$$

and $$m\widehat{BC} = m\widehat{BD} + m\widehat{DC}.$$

(Why, in each case?) By Case 1, we know that

$$m\angle v = \frac{1}{2}\, m\widehat{BD}$$

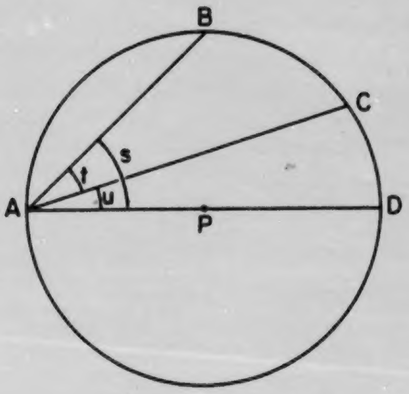
and $$m\angle w = \frac{1}{2}\, m\widehat{DC}.$$

Putting these equations together, we get

$$\begin{aligned} m\angle A &= \frac{1}{2}\, m\widehat{BD} + \frac{1}{2}\, m\widehat{DC} \\ &= \frac{1}{2}\, m\widehat{BC}. \end{aligned}$$

which was to be proved.

Case 3. Suppose that B and C are on the same side of the diameter through A, like this:



The proof here is very much like that for Case 2, and we state it in condensed form:

$$\begin{aligned} m\angle BAC = m\angle t &= m\angle s - m\angle u \\ &= \frac{1}{2}\, m\widehat{BD} - \frac{1}{2}\, m\widehat{CD} \\ &= \frac{1}{2}(m\widehat{BD} - m\widehat{CD}) \\ &= \frac{1}{2}\, m\widehat{BC}. \end{aligned}$$

You should check carefully to make sure that you see why each of these equations is correct.

From this theorem we get two very important corollaries:

Corollary 13-7-1. An angle inscribed in a semi-circle is a right angle.

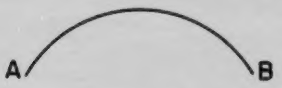
This is so because such an angle intercepts a semi-circle, which has measure 180.

Corollary 13-7-2. Angles inscribed in the same arc are congruent.

The proof of this is fairly obvious because all such angles intercept the same arc.

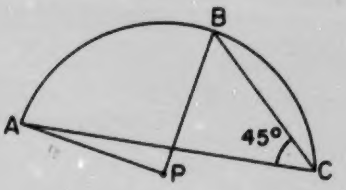
Problem Set 13-4a

1. The center of an arc is the center of the circle of which the arc is a part. How would you find the center of $\widehat{AB}$?



2. Given: P is the center of $\widehat{AC}$, $m\angle C = 45$.

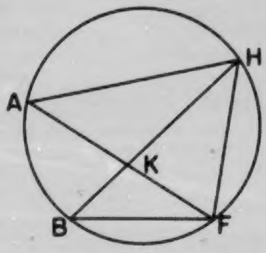
   Prove: $\overline{BP} \perp \overline{AP}$.



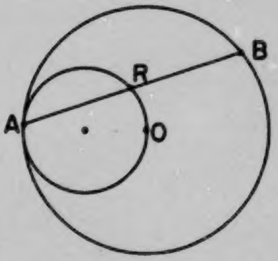
3. In the figure, $m\widehat{AB} = m\widehat{BF}$.

   a. Prove $\Delta\ AHK \sim \Delta\ BHF$.

   b. What other triangle is similar to $\Delta\ BHF$?

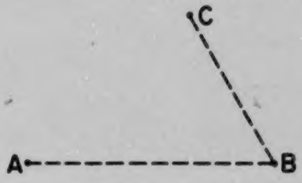


4. The two circles in this figure are tangent at A and the smaller circle passes through O, the center of the larger circle. Prove that any chord of the larger circle with end-point A is bisected by the smaller circle.

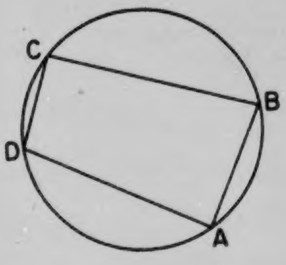


*5. Prove: Any three non-collinear points lie on a circle.

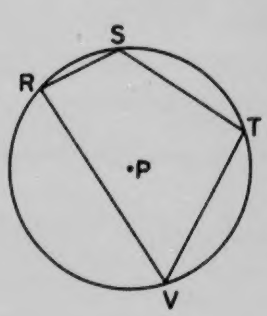
Restatement: A, B, and C are non-collinear. Prove that there is a circle containing A, B, and C. (Hint: Draw $\overline{AB}$ and $\overline{BC}$. Can you find the center of the circle?)



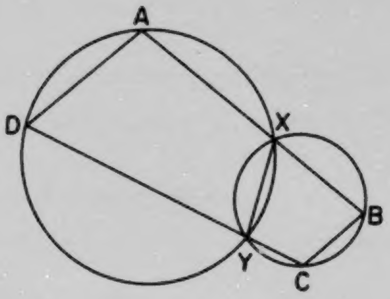
6. An inscribed quadrilateral is a quadrilateral having all of its vertices on a circle. Prove the theorem: The opposite angles of an inscribed quadrilateral are supplementary.



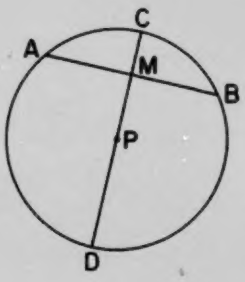
7. In circle P, let $m\angle R = 85$, $m\widehat{RS} = 40$, $m\widehat{TV} = 90$. Find the measures of the other arcs and angles in the figure.



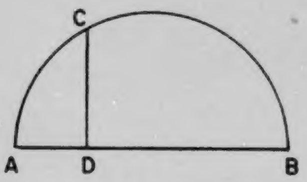
8. $\overline{XY}$ is the common chord of two intersecting circles. $\overline{AB}$ and $\overline{DC}$ are two segments cutting the circles as shown in the figure and containing X and Y respectively.
Prove: $\overline{AD} \parallel \overline{BC}$.
(Hint: See Problem 6.)



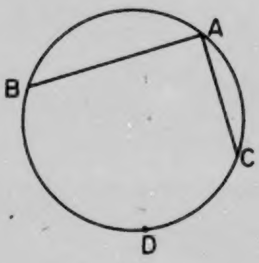
9. Prove: A diameter perpendicular to a chord of a circle bisects both arcs determined by the chord.



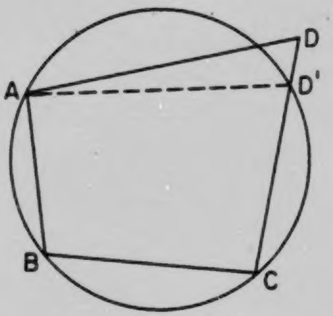
10. In the figure, $\overset{\frown}{ACB}$ is a semi-circle and $\overline{CD} \perp \overline{AB}$. Prove that CD is the geometric mean of AD and BD.



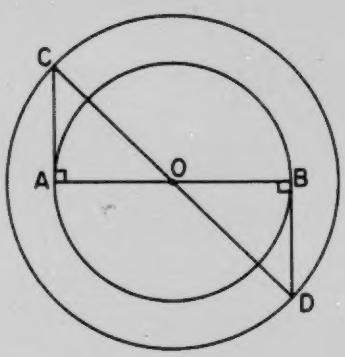
11. Prove the following converse of Corollary 13-7-1: If an angle inscribed in a circular arc is a right angle, then the arc is a semi-circle.



*12. If a pair of opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.
(Hint: Use Problems 5 and 6 in an indirect proof.)



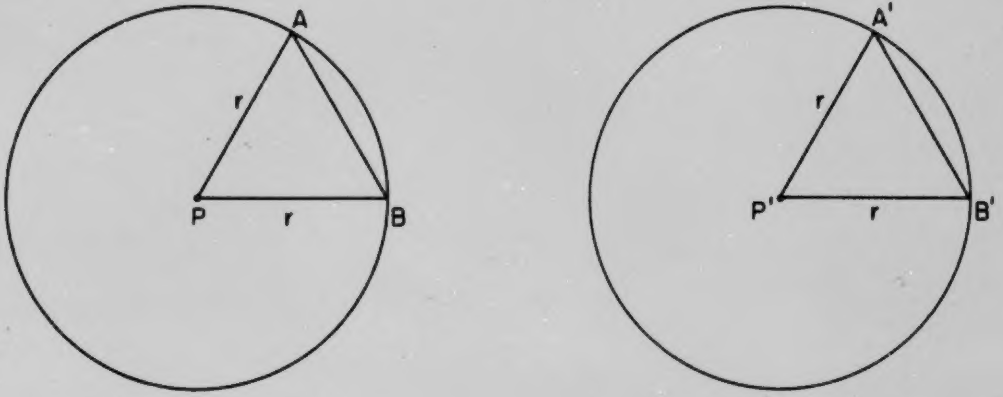
*13. In this figure, $\overline{AB}$ is a diameter of the smaller of two concentric circles, both with center O, and $\overline{AC}$ and $\overline{BD}$ are tangent to the smaller circle. $\overline{CO}$ and $\overline{DO}$ are radii of the larger circle. Prove that $\overline{CD}$ is a diameter of the larger circle.
(Hint: Draw $\overline{AD}$ and $\overline{CB}$.)



Definition: In the same circle, or in congruent circles, two arcs are called congruent if they have the same measure.

Just as in the definition of congruent segments, angles, triangles or circles, the intuitive idea is that one arc can be moved so as to coincide with the other.

Theorem 13-8. In the same circle or in congruent circles, if two chords are congruent, then so also are the corresponding minor arcs.



Proof: We need to show, in the above figure, that if $AB = A'B'$, then $\widehat{AB} \cong \widehat{A'B'}$. By the S.S.S. Theorem, we have

$$\Delta\ APB \cong \Delta\ A'B'P'.$$

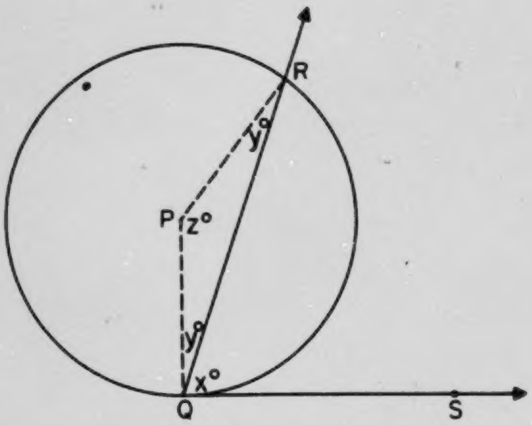
Therefore $\angle P \cong \angle P'$. Since $m\widehat{AB} = m\angle P$ and $m\widehat{A'B'} = m\angle P'$, this means that $\widehat{AB} \cong \widehat{A'B'}$, which was to be proved.

The converse is also true, and the proof is very similar:

Theorem 13-9. In the same circle or in congruent circles, if two arcs are congruent, then so are the corresponding chords.

That is, in the figure above, if $\widehat{AB} = \widehat{A'B'}$, then $AB = A'B'$. And if it is the major arcs that are known to be congruent, then the same conclusion holds.

Theorem 13-10. Given an angle with vertex on the circle formed by a secant ray and a tangent ray. The measure of the angle is half the measure of the intercepted arc.



Proof: By the angle formed by a secant ray and tangent ray we mean the angle as illustrated in the figure above. We prove the theorem for the case in which the angle is acute, as in the figure. We use the notation of the figure for the measures of the various angles. In Δ PQR, $\angle$ R and $\angle$ Q have the same measure y, as indicated, because Δ PQR is isosceles. Since $m\widehat{QR} = m\angle QPR$, what we need to prove is that $x = \frac{1}{2}z$.

By Corollary 13-2-1, $\angle$ PQS is a right angle. Therefore

$$x = 90 - y.$$

By Theorem 9-13, $z + y + y = 180$, so that

$$z = 180 - 2y.$$

Therefore $x = \frac{1}{2}z$, which was to be proved.

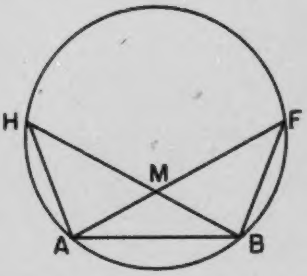
Problem Set 13-4b

1. Prove Theorem 13-9: In the same circle or in congruent circles, if two arcs are congruent, then so are the corresponding chords.

2. In the figure $AF = BH$.

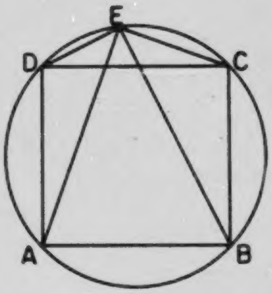
   Prove: a. $\widehat{AH} \cong \widehat{FB}$.

   b. $\triangle AMH \cong \triangle BMF$.



3. ABCD is an inscribed square. E is any point of $\widehat{DC}$, as shown in this figure.

   Prove that $\overline{AE}$ and $\overline{BE}$ trisect $\angle DEC$.



4. In the figure, A, B, C, D are on the circle and $\overleftrightarrow{EF}$ is tangent to the circle at A. Complete the following statements:

   a. $\angle BDC \cong$ ______.

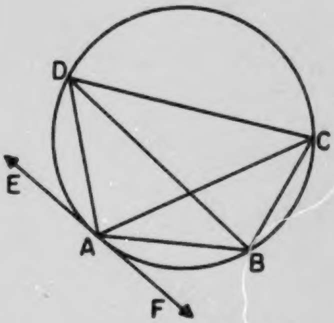
   b. $\angle ADC \cong$ ______.

   c. $\angle ACB \cong$ ______ $\cong$ ______.

   d. $\angle EAD$ is supplementary to ______.

   e. $\angle DAB$ is supplementary to ______.

   f. $\angle ABC$ is supplementary to ______.



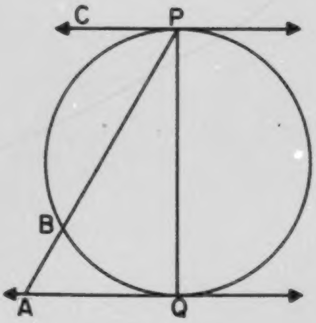
g. $\angle DAE \cong$ _____ $\cong$ _____.

h. $\angle DBA$ is supplementary to _____.

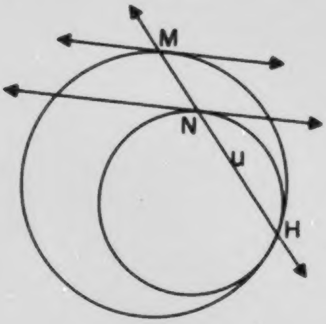
i. $\angle ADB$ is supplementary to _____.

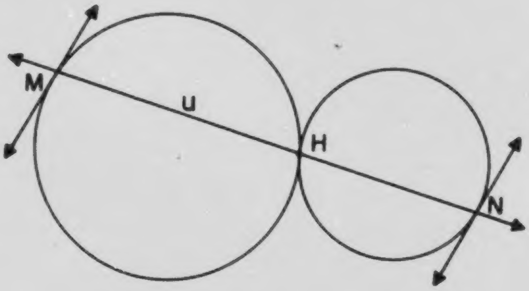
j. $\angle DAC \cong$ _____.

5. In the figure $\overleftrightarrow{CP}$ and $\overleftrightarrow{AQ}$ are tangents, $\overline{PQ}$ is a diameter of the circle. If $m\widehat{PB} = 120$ and the radius of the circle is 3, find the length of $\overline{AP}$.



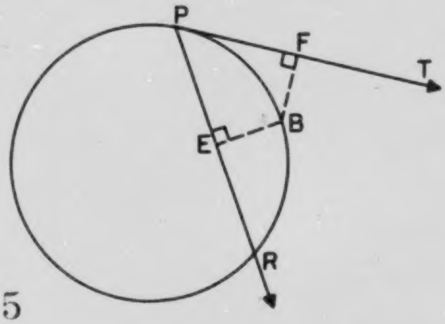
*6. Two circles are tangent, either internally or externally, at a point H. Let u be any line through H meeting the circles again at M and N. Prove that the tangents at M and N are parallel.



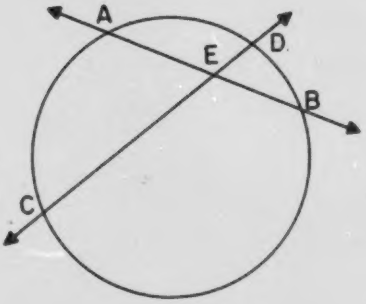


*7. Given: Tangent $\overleftrightarrow{PT}$ and secant $\overleftrightarrow{PR}$. B is the midpoint of $\widehat{PR}$.

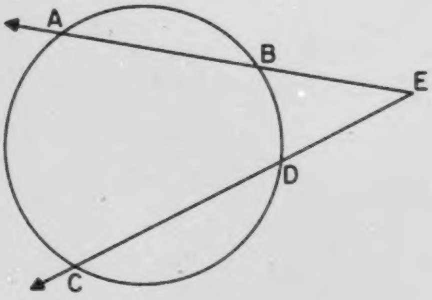
Prove: B is equidistant from $\overleftrightarrow{PT}$ and $\overleftrightarrow{PR}$.



8. Prove the theorem: The measure of an angle formed by two secants of a circle intersecting in the interior of a circle is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.
Given: A circle with secants $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersecting at E.
Prove: $m\angle DEB = \frac{1}{2}(m\widehat{DB} + m\widehat{AC})$.
(Hint: Draw $\overline{BC}$.)

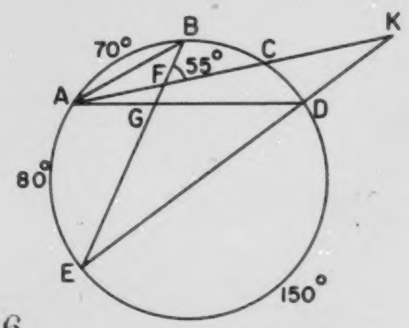


9. Prove the theorem: The measure of an angle formed by two secants of a circle meeting in the exterior of the circle is one-half the difference of the measures of the intercepted arcs. (Hint: See Problem 8.)

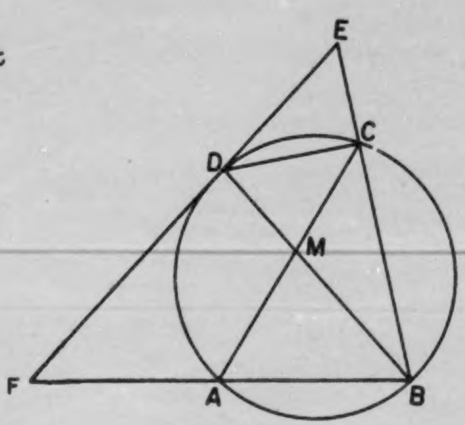


10. Verify that the theorem of Problem 9 holds if the words "two secants" are replaced by "a secant and a tangent" or by "two tangents."

11. In the figure, let $m\widehat{AB} = 70$, $m\widehat{AE} = 80$, $m\widehat{ED} = 150$, and $m\angle BFC = 55$.
Find $m\widehat{BC}$, $m\widehat{CD}$, $m\angle K$, $m\angle E$, $m\angle BAD$, $m\angle AGE$, $m\angle DGE$, $m\angle ADK$.



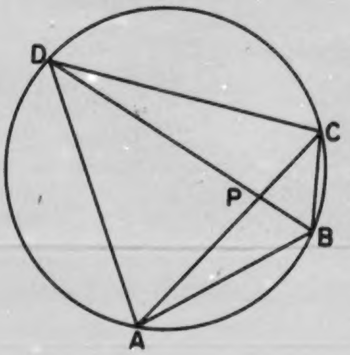
12. In the figure, $\overline{EF}$ is tangent to the circle at D and $\overline{AC}$ bisects $\angle BCD$. If $m\overset{\frown}{AB} = 88$ and $m\overset{\frown}{CD} = 62$, find the measure of each arc and each angle of the figure.



13. Given inscribed quadrilateral ABCD with diagonals intersecting at P.

Prove: a. $\triangle APD \sim \triangle BPC$.

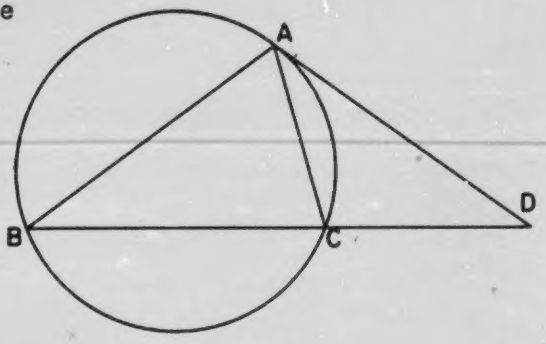
b. $AP \cdot PC = PD \cdot PB$.



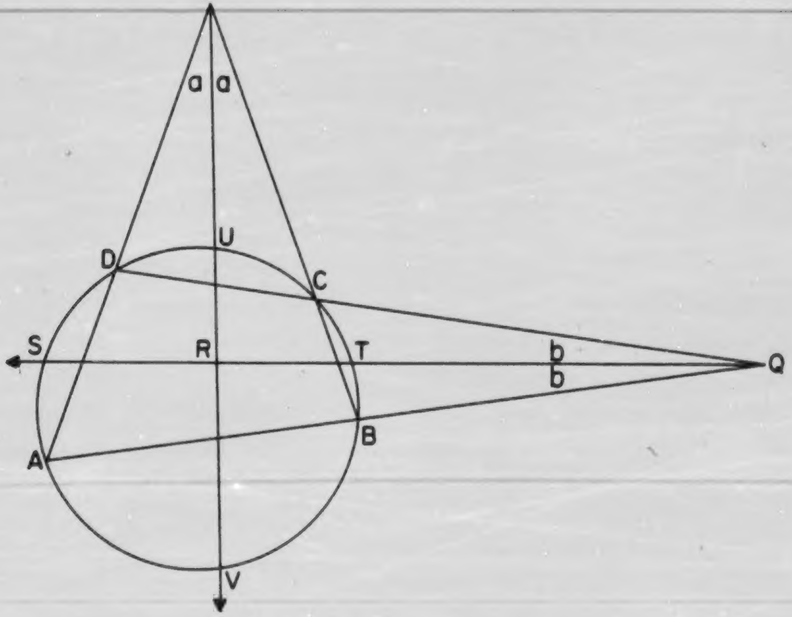
14. Given $\overleftrightarrow{AD}$ tangent to the circle at A and secant $\overleftrightarrow{BD}$ intersecting the circle at B and C.

Prove: a. $\triangle ABD \sim \triangle CAD$.

b. $BD \cdot CD = AD^2$.

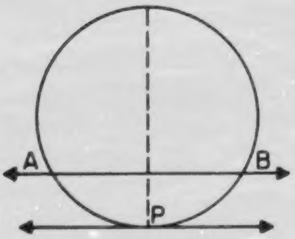


*15. In the figure, quadrilateral ABCD is inscribed in the circle; lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect in P, lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ intersect in Q; $\overrightarrow{PV}$ and $\overrightarrow{QS}$ are the bisectors of $\angle APB$ and $\angle AQD$ respectively.
Prove: $\overleftrightarrow{PV} \perp \overleftrightarrow{QS}$.

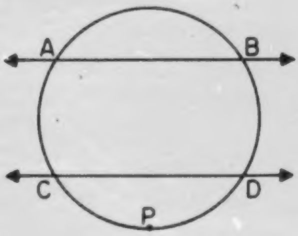


(Hint: Show $m\angle PRQ = m\angle QRV$. Use theorems developed in this Problem Set.)

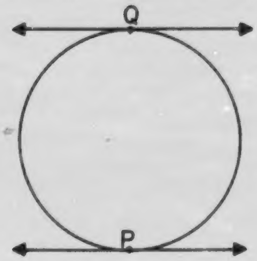
*16. Prove the theorem: If two parallel lines intersect a circle, they intercept congruent arcs.



Case I
(One tangent - one secant)



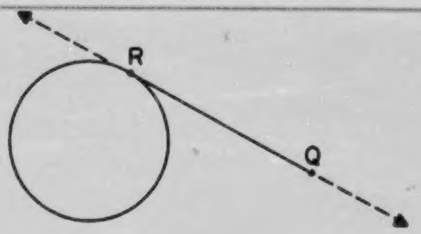
Case II
(Two secants)



Case III
(Two tangents)

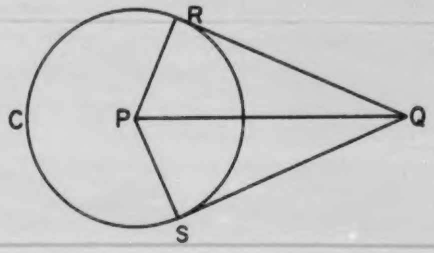
13-5. Lengths of Tangent and Secant Segments.

Definition: If the line $\overleftrightarrow{QR}$ is tangent to a circle at R, then the segment $\overline{QR}$ is a tangent segment from Q to the circle.



Theorem 13-11. The two tangent segments to a circle from an external point are congruent, and form congruent angles with the line joining the external point to the center of the circle.

Restatement: If $\overline{QR}$ is tangent to the circle C at R, and $\overline{QS}$ is tangent to C at S, then $\overline{QR} \cong \overline{QS}$, and $\angle PQR \cong \angle PQS$.

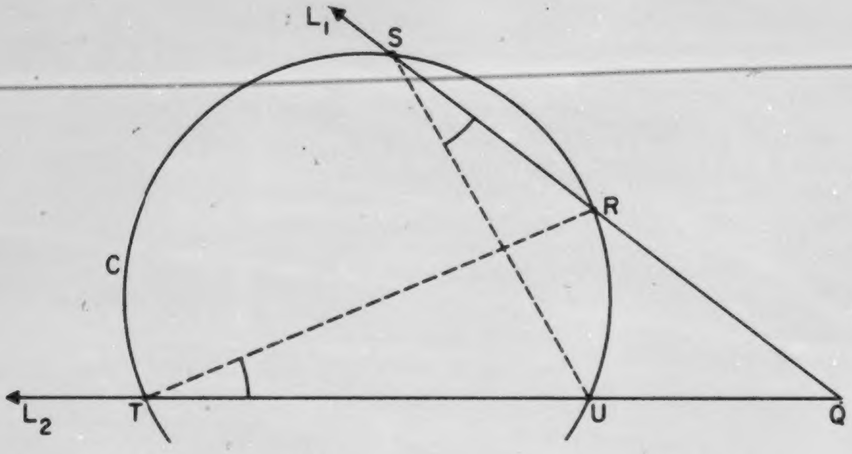


Proof: By Corollary 13-2-1, ΔPQR and ΔPQS are right triangles, with right angles at R and S. Obviously $PQ = PQ$ and $PR = PS$ because R and S are points of the circle. By the Hypotenuse-Leg Theorem (Theorem 7-3), this means that

$$\Delta PQR \cong \Delta PQS.$$

Therefore $\overline{QR} \cong \overline{QS}$, and $\angle PQR \cong \angle PQS$, which was to be proved.

The statement of the following theorem is easier to understand if we look at a figure first:



The theorem says that given any two secant lines through Q, as in the figure, we have

$$QR \cdot QS = QU \cdot QT.$$

Theorem 13-12. Given a circle C and an external point Q, let L_1 be a secant line through Q, intersecting C in points R and S; and let L_2 be another secant line through Q, intersecting C in points T and U. Then $QR \cdot QS = QU \cdot QT$.

Proof: Consider the triangles $\Delta\ SQU$ and $\Delta\ TQR$. These triangles have $\angle Q$ in common. And $\angle S \cong \angle T$, as indicated in the figure, because both of these angles are inscribed in the major arc $\widehat{RU}$. By the A.A. Corollary (Corollary 12-3-1), this means that

$$\Delta\ SQU \sim \Delta\ TQR.$$

Therefore corresponding sides are proportional. Hence

$$\frac{QS}{QT} = \frac{QU}{QR},$$

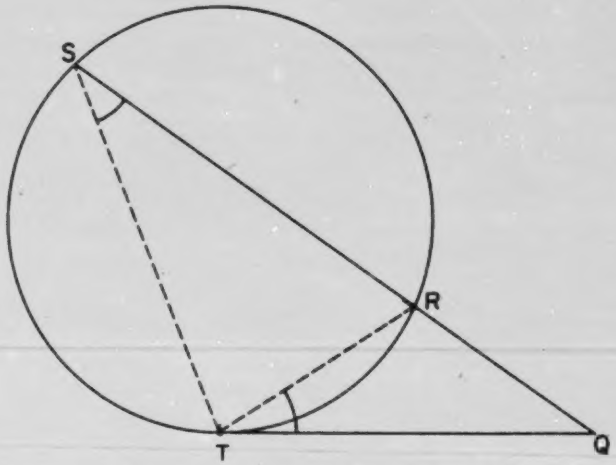
and

$$QR \cdot QS = QU \cdot QT,$$

which was to be proved.

Notice that this theorem means that the product $QR \cdot QS$ is determined merely by the given circle and the given external point, and is independent of the choice of the secant line. (The theorem tells us that any other secant line gives the same product.) This constant product is called the power of the point with respect to the circle.

The following theorem is going to say that in the figure below, $QR \cdot QS = QT^2$.



Theorem 13-13. Given a tangent segment $\overline{QT}$ to a circle, and a secant line through Q, intersecting the circle in points R and S. Then

$$QR \cdot QS = QT^2.$$

The main steps in the proof are as follows. You should find the reasons in each case.

(1) $m\angle S = \frac{1}{2} m\widehat{TR}$.

(2) $m\angle RTQ = \frac{1}{2} m\widehat{TR}$.

(3) $\angle S \cong \angle RTQ$.

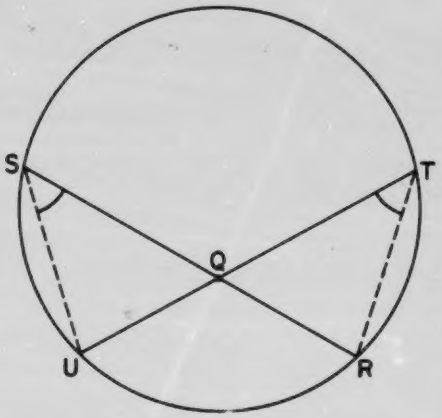
(4) $\Delta QRT \sim \Delta QTS$.

(5) $\frac{QR}{QT} = \frac{QT}{QS}$.

(6) $QR \cdot QS = QT^2$.

The following theorem is a further variation on the preceding two; the difference is that now we are going to draw two lines through a point in the interior of the circle. The theorem says that in the figure below, we always have

$$QR \cdot QS = QU \cdot QT.$$



The main steps in the proof are as follows: You should find the reason in each case:

(1) $\angle S \cong \angle T$.

(2) $\angle SQU \cong \angle TQR$.

(3) $\Delta SQU \sim \Delta TQR$.

(4) $\frac{QS}{QT} = \frac{QU}{QR}$.

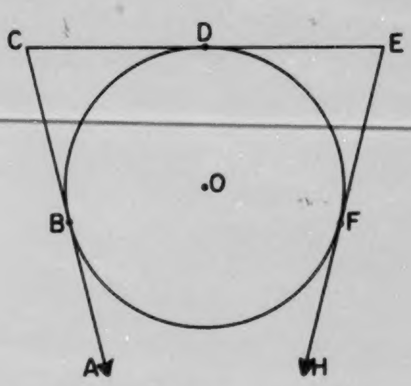
(5) $QR \cdot QS = QU \cdot QT$.

For purposes of reference, let us call this Theorem 13-14. Write a complete statement of the theorem. That is, write a statement that can stand alone, without a figure.

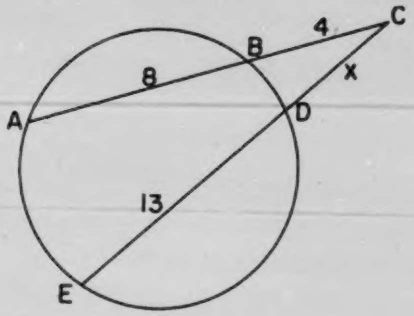
Problem Set 13-5

1. $\overleftrightarrow{AC}$, $\overleftrightarrow{CE}$ and $\overleftrightarrow{EH}$ are tangent to circle O at B, D, and F respectively.

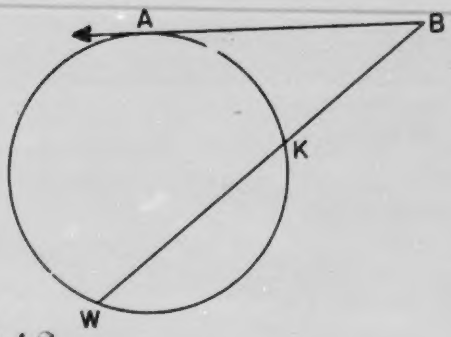
Prove: CB + EF = CE.



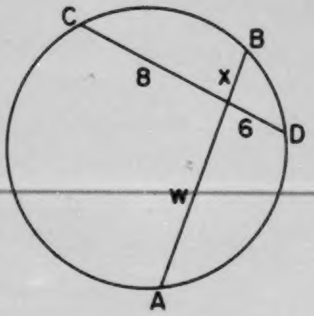
2. Secants $\overleftrightarrow{CA}$ and $\overleftrightarrow{CE}$ intersect the circle at A, B, and D, E, as given in this figure. If the lengths of the segments are as shown, find x.



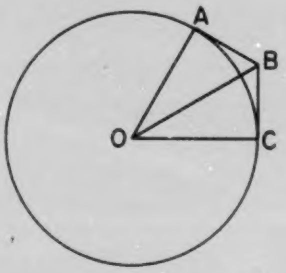
3. In this figure $\overleftrightarrow{AB}$ is tangent to the circle at A and secant $\overleftrightarrow{BW}$ intersects the circle at K and W. If AB = 6 and WK = 5, how long is $\overline{BK}$?



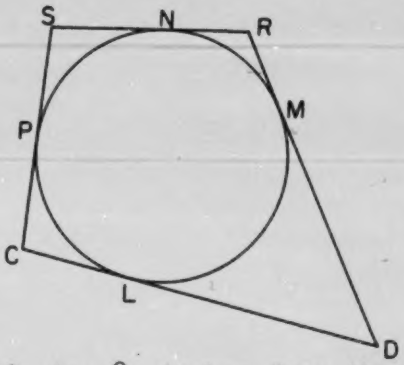
4. Given a circle with intersecting chords as shown and with $x < w$, if $AB = 19$, find x and w.



5. $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are tangent to circle O at A and C, respectively, and $m\angle ABC = 120$. Prove that $AB + BC = OB$.

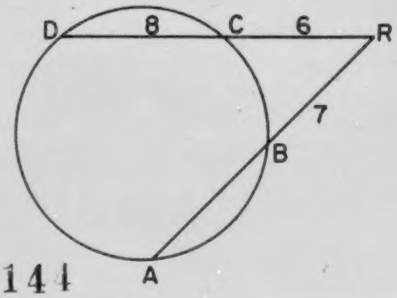


6. Given: The sides of quadrilateral CDRS are tangent to a circle at L, M, N, P as in the figure.
Prove: $SR + CD = SC + RD$.

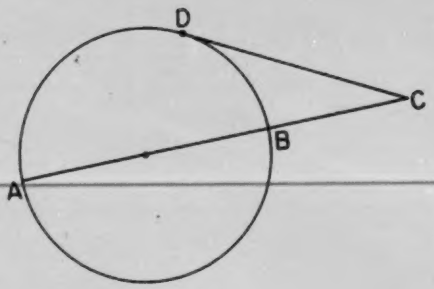


7. In a circle a chord of length 12 is 8 inches from the center of the circle. Using Theorem 13-14, find the radius of the circle.

8. Secants and segments are as indicated. Find the length of $\overline{AB}$.

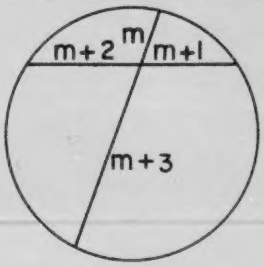


9. In the figure, $\overline{CD}$ is a tangent segment to the circle at D and $\overline{AC}$ is a segment of a secant which contains the center of the circle. If CD = 12 and CB = 4, find the radius of the circle.

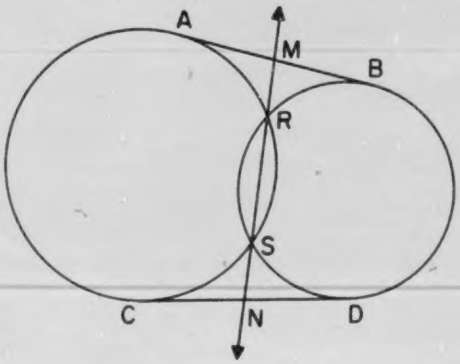


10. If two tangent segments to a circle form an equilateral triangle with the chord having the points of tangency as its end-points, find the measure of each arc of the chord.

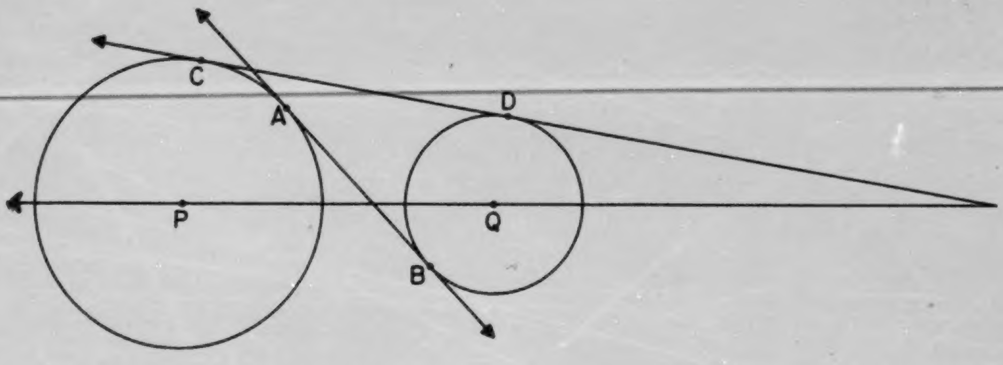
11. Show that it is not possible for the lengths of the segments of two intersecting chords to be four consecutive integers.



*12. Prove that if two circles intersect, the common secant bisects either common tangent segment.



13. If a common tangent of two circles meets the line of centers at a point between the centers it is called a common internal tangent. If it does not meet the line of centers at a point between the centers it is called a common external tangent.



In the figure $\overleftrightarrow{AB}$ is a common internal tangent and $\overleftrightarrow{CD}$ is a common external tangent.

a. In the figure above, how many common tangents are possible? Specify how many of each kind.

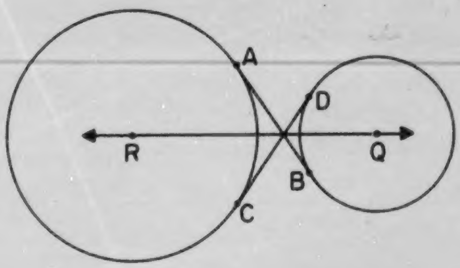
b. If the circles were externally tangent, how many tangents of each kind?

c. If the circles were intersecting at two points.

d. If the circles were internally tangent?

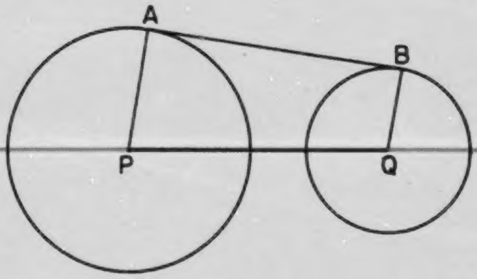
e. If the circles were concentric?

*14. Prove: The common internal tangents of two circles meet the line of centers at the same point. (Hint: Use an indirect proof.)



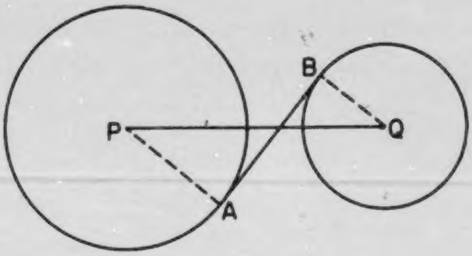
*15. Prove that the common tangent segments of common internal tangents are congruent. Use figure of Problem 14.

16. The radii of two circles have lengths 22 and 8 respectively and the distance between their centers is 50. Find the length of the common external tangent segment. (Hint: Draw a perpendicular through Q to $\overline{AP}$.)

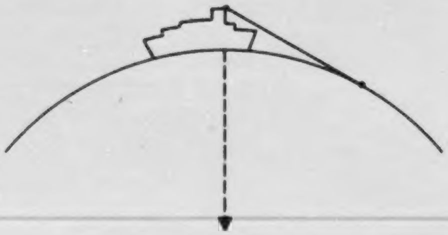


17. Two circles have a common external tangent segment 36 inches long. Their radii are 6 inches and 21 inches respectively. Find the distance between their centers.

18. The distance between the centers of two circles having radii of 7 and 9 is 20. Find the length of the common internal tangent segment.



*19. Standing on the bridge of a large ship on the ocean, the captain asked a new young officer to determine the distance to the horizon. The young officer took a pencil and paper and in a few moments came up with an answer. On the paper he had written the formula $d = \frac{5}{4}\sqrt{h}$ miles. Show that this formula is correct approximately where h is the height in feet of the observer above the water and d is the distance in miles to the horizon. (Assume the diameter of the earth to be 8000 miles.)



Review Problems

1. For circle O,

   a. $\overline{BC}$ is a ______.

   b. $\overline{AD}$ is a ______.

   c. $\overleftrightarrow{AC}$ is a ______.

   d. $\overline{OA}$ is a ______.

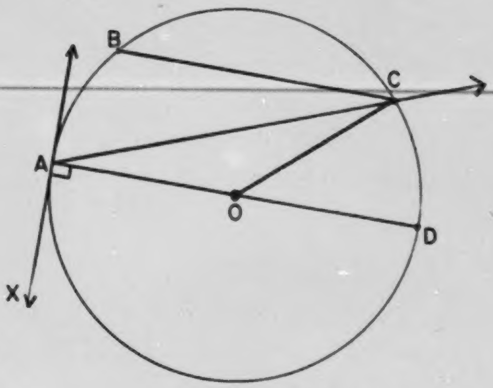
   e. $\overleftrightarrow{AX}$ is a ______.

   f. $\overset{\frown}{CD}$ is a _____ _____.

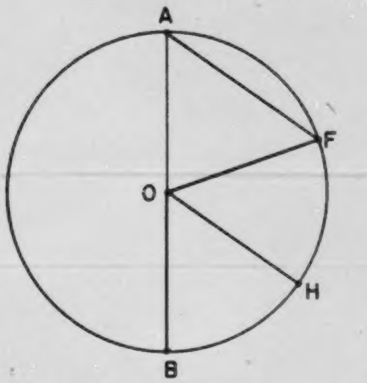
   g. $\overset{\frown}{ADC}$ is a _____ _____.

   h. $\angle BCA$ is an _____ _____.

   i. $\angle COD$ is a _____ _____.



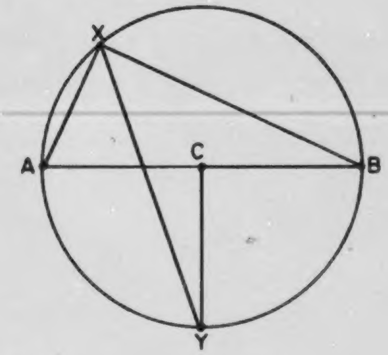
2. Given: In the figure, circle O has diameter $\overline{AB}$. $\overline{AF} \parallel \overline{OH}$, $m\angle A = 55$. Find $m\overset{\frown}{BH}$ and $m\overset{\frown}{AF}$.



3. Given: $\overline{AB}$ is a diameter of circle C. $\overrightarrow{XY}$ bisects $\angle AXB$.

   Prove: $\overline{CY} \perp \overline{AB}$.

   (Hint: Find $m\angle AXY$.)



4. Indicate whether each of the following statements is true or false.

   a. If a point is the mid-point of two chords of a circle, then the point is the center of the circle.

   b. If the measure of one arc of a circle is twice the measure of a second arc, then the chord of the second arc is less than twice as long as the chord of the first arc.

   c. A line which bisects two chords of a circle is perpendicular to each of the chords.

   d. If the vertices of a quadrilateral are on a circle, then each two of its opposite angles are supplementary.

   e. If each of two circles is tangent to a third circle, then the two circles are tangent to each other.

   f. A circle cannot contain three collinear points.

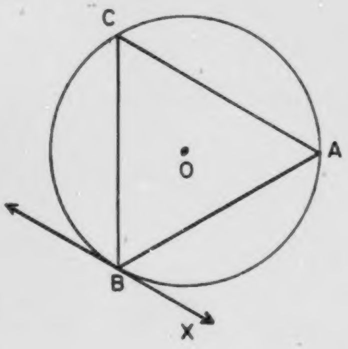
   g. If a line bisects a chord of a circle, then it bisects the minor arc of that chord.

   h. If $\overline{PR}$ is a diameter of circle O and Q is any point in the interior of circle O not on $\overline{PR}$, then $\angle PQR$ is obtuse.

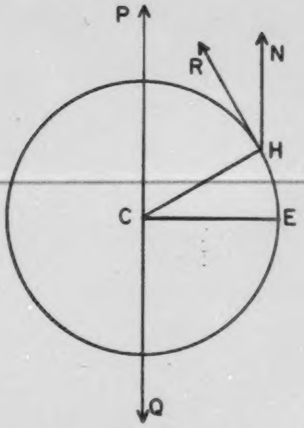
   i. A tangent to a circle at the mid-point of an arc is parallel to the chord of that arc.

   j. It is possible for two tangents to the same circle to be perpendicular to each other.

5. Given: In the figure $\overleftrightarrow{BX}$ is tangent to circle O at B. AB = AC. $m\overset{\frown}{CB} = 100$. Find $m\angle C$ and $m\angle ABX$.



6. Given: Circle C with $\overline{EC} \perp \overleftrightarrow{PQ}$, $\overrightarrow{HN} \parallel \overleftrightarrow{PQ}$, and $\overrightarrow{HR}$ tangent to circle C at H.
Prove: $m\widehat{HE} = m\angle RHN$.
(Note: The circle may be considered to represent the earth, with $\overleftrightarrow{PQ}$ the earth's axis, $\angle RHN$ the angle of elevation of the North Star, and $m\widehat{HE}$ the latitude of a point H.)

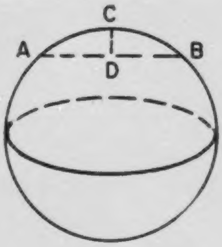


7. A hole 40 inches in diameter is cut in a sheet of plywood, and a sphere 50 inches in diameter is set in this hole. How far below the surface of the board will the globe sink?

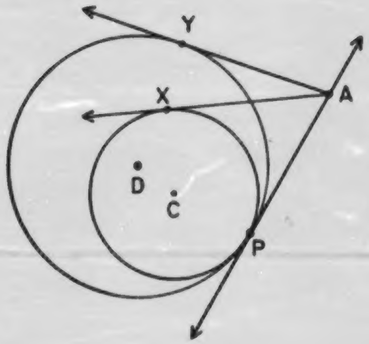
8. A wheel is broken so that only a portion of the rim remains. In order to find the diameter of the wheel the following measurements are made: three points C, A, and B are taken on the rim so that chord $\overline{AB} \cong$ chord $\overline{AC}$. The chords $\overline{AB}$ and $\overline{AC}$ are each 15 inches long, and the chord $\overline{BC}$ is 24 inches long. Find the diameter of the wheel.

9. Diameter $\overline{AD}$ of circle C contains a point B which lies between A and C. Prove that $\overline{BA}$ is the shortest segment joining B to the circle and $\overline{BD}$ is the longest.

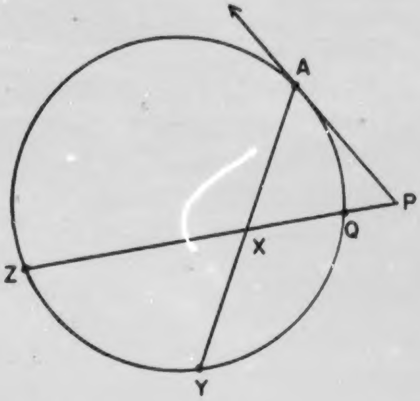
*10. Assume that the earth is a sphere of radius 4,000 miles. A straight tunnel $\overline{AB}$ 200 miles long connects two points A and B on the surface, and a ventilation shaft $\overline{CD}$ is constructed at the center of the tunnel. What is the length (in miles) of this shaft?



11. Given: Circles C and D internally tangent at P with common tangent $\overleftrightarrow{AP}$. $\overleftrightarrow{AX}$ is tangent to circle C at X and $\overleftrightarrow{AY}$ is tangent to circle D at Y.
Prove: AY = AX.



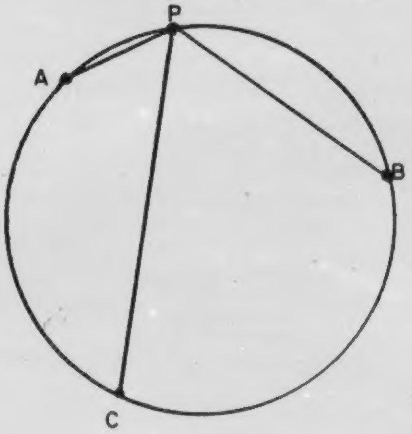
*12. In the figure, $\overleftrightarrow{AP}$ is tangent to the circle at A. AP = PX = XY. If PQ = 1 and QZ = 8 find AX.



*13. Given: $\overset{\frown}{AB}$, $\overset{\frown}{BC}$ and $\overset{\frown}{CA}$ are 120° arcs on a circle and P is a point on $\overset{\frown}{AB}$.

Prove: PA + PB = PC.

(Hint: Consider a parallel to $\overline{PB}$ through A intersecting $\overline{PC}$ in R and the circle in Q.)



Chapter 14

CHARACTERIZATION OF SETS. CONSTRUCTIONS.

14-1. Characterization of Sets.

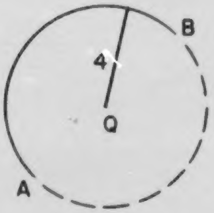
In Chapter 6 we showed how a certain figure, the perpendicular bisector of a segment, could be specified in terms of a characteristic property of its points, namely, that each of them is equidistant from the end-points of the segment.

In Chapter 13 a circle (and a sphere) was defined in terms of a characteristic property of its points, namely, that each of them is at a given distance from the center.

Such characterizations or descriptions of a point set (geometric figure) in terms of a common property of its points are often very useful, and we shall spend some time discussing them.

What do we mean when we say that a set is characterized by a condition, or a set of conditions, imposed on its points? In the first place, we certainly mean that every point of the set satisfies the conditions. But this is not enough, as we can readily see from an example. Suppose the condition is "in plane E at distance 4 from point Q in E". A semi-circle in E with center Q and radius 4 has all its points satisfying this condition. So does any other suitable arc.

Every point in $\widehat{AB}$ is 4 units distant from Q, but not every point 4 units distant from Q is in $\widehat{AB}$.



The obvious trouble with such examples is that they leave out some points that satisfy the conditions. We want the whole circle, not just a part of it. In general, we want our set to contain all points that satisfy the conditions. Another way of saying this is that every point that satisfies the conditions is a point of the set. This is the second part of the meaning of characterization.

Let us put the two parts together for future reference:

(1) Every point of the set satisfies the conditions,

(2) Every point which satisfies the conditions is a point of the set.

If you refer to Theorem 6-2, you will see that the restatement of this theorem is worded in exactly this form.

Problem Set 14-1

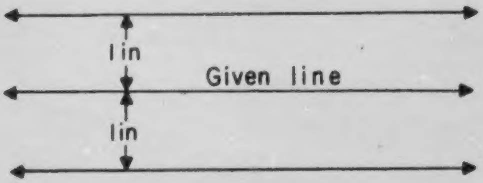
These problems are proposed for discussion. No proofs are expected. In some of the problems in this set we speak of the distance from a point to a figure. This is defined as the shortest distance from the point to any point of the figure.

Illustrative example: Describe and sketch the set of points which are **one** inch from a given line.

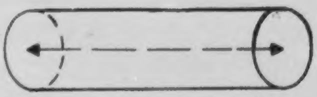
a. In a plane.

b. In space.

Answer:



a. The set consists of two lines, each one inch from the given line and parallel to it.



b. The set consists of all points of a cylindrical surface with one inch radius and the given line as axis.

1. What set of points P is characterized by the condition that $CP = 3$ inches, where C is a given point?

2. What set of points P in a given plane E is characterized by the condition that $CP = 3$ inches, where C is a given point of E?

3. Describe and sketch the set of points in a plane E which are equidistant from each of two parallel lines in E.

4. E is a plane and C is a fixed point 3 inches from the plane. What is the set of points in E whose distance from C is

   a. 5 inches?

   b. 3 inches?

   c. 2 inches?

5. E is a plane. L and M are two intersecting lines in E.

   a. How many points of E are 2 inches from L and 2 inches from M?

   b. Sketch the set of points of E whose distances from L and M are each at most one inch.

6. E is a plane. A and B are two points in E which are 4 feet apart. What is the set of points of E which are

   a. 4 feet from A and 4 feet from B?

   b. At most 4 feet from A and at most 4 feet from B?

   c. 2 feet from A and 2 feet from B?

   d. 1 foot from A and 1 foot from B?

7. $\overline{AB}$ is a segment of length 3 inches in a plane E. Describe and sketch the set of those points of E which are one inch from $\overline{AB}$.

14-2. Basic Characterizations. Concurrence Theorems.

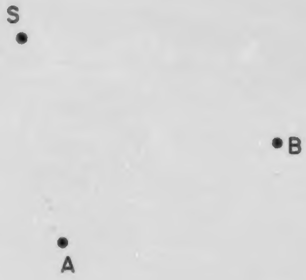
For convenience in reference we restate here some of the characterizations we have already met. Some of these are definitions and some are theorems.

1. A sphere is the set of points at a given distance from a given point.
2. A circle is the set of points in a given plane at a given distance from a given point of the plane.
3. The perpendicular bisecting plane of a given segment is the set of points equidistant from the end-points of the segment.
4. The perpendicular bisector, in a given plane, of a given segment in the plane, is the set of points in the plane equidistant from the end-points of the segment.

Problem Set 14-2a

1. Describe the set of points at a given distance from
   - a. a given point.
   - b. a given line.
   - c. a given plane.
   - d. each of two intersecting planes.
   - e. each of two given points.
   - f. a segment.
2. Describe the set of points in a plane equidistant from
   - a. two points.
   - b. two parallel lines.
   - c. two intersecting lines.
   - d. three non-collinear points.

3. Describe the set of points equidistant from
    - a. two given points.
    - b. two parallel lines.
    - c. two parallel planes.
    - d. two intersecting planes.
    - e. a plane and a line perpendicular to it.
4. Indicate whether each statement is true or false.
    - a. Given a line u and a plane E there is always a plane
        - 1. containing u and perpendicular to E.
        - 2. containing u and parallel to E.
    - b. Given two non-intersecting lines in space, there is always a plane containing one and
        - 1. parallel to the other.
        - 2. perpendicular to the other.
5. The Smiths, the Allens and the Browns live in homes represented by these three points. They plan to erect a flagpole at a point which will be equidistant from their back doors. Tell how to find the point where they should place the pole.



6. Describe the set which consists of the vertices of all isosceles triangles having $\overline{AB}$ as base.

7. Find a point in the plane equally distant from three non-collinear points. Why must the points be non-collinear?

8. What is the set of points which are equidistant from two given points and at the same time equidistant from two given parallel planes? (Hint: Consider the intersection of the set of points representing the separate conditions. There may be more than one solution depending on the positions of the given elements.)

*9. What is the set of points in a plane which are within four centimeters of one or the other of two points in a plane which are four centimeters apart?

10. Let L and M be any two intersecting lines. Choose any two coordinate systems on these lines (not necessarily with 0 at the point of intersection). Draw a number of lines through corresponding points; that is, points with the same coordinates. For example, see Figure A.

If you put in enough lines, the figure should appear to include a nearly smooth curve. Experiment with this construction, trying different pairs of lines and different coordinate systems.

The construction is quite general, but some choices of coordinate systems on the two lines will lead to more satisfying results on your paper than others.

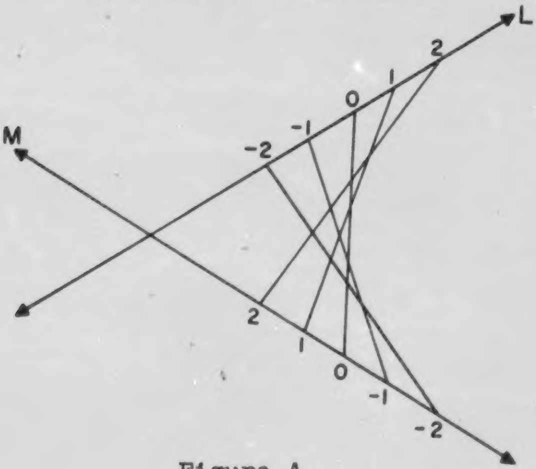


Figure A.

11. What is the set of points in a plane at a given distance from a square of side 2 in the plane? Consider the three cases $d > 1$, $d = 1$, $d < 1$.

*12. F and G are two points in a plane E. $FG = 4$. Sketch the set of those points P of E, such that $PF + PG = 5$.

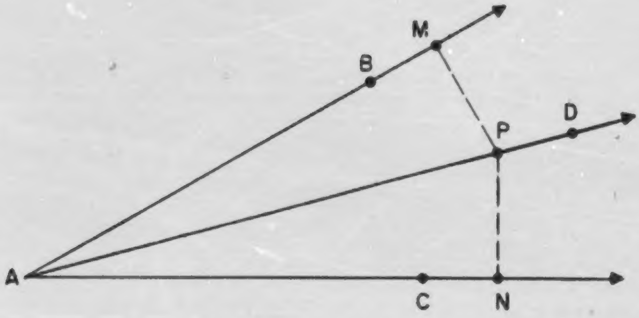
Another characterization you can include in the above list is the following theorem:

Theorem 14-1. The bisector of an angle, minus its end-point, is the set of points in the interior of the angle equidistant from the sides of the angle.

Restatement: Let $\overrightarrow{AD}$ bisect $\angle BAC$.

(1) If P is on $\overrightarrow{AD}$ but $P \neq A$, then P is in the interior of $\angle BAC$ and the distance from P to $\overleftrightarrow{AB}$ equals the distance from P to $\overleftrightarrow{AC}$.

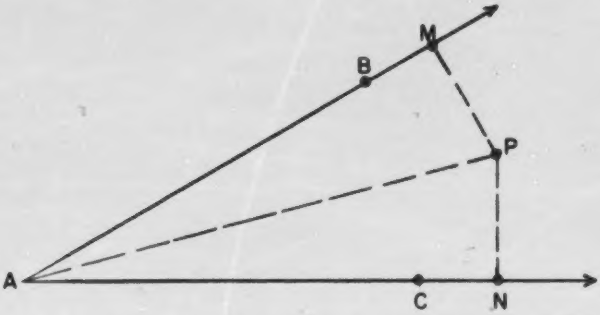
(2) If P is in the interior of $\angle BAC$ and the distance from P to $\overleftrightarrow{AB}$ equals the distance from P to $\overleftrightarrow{AC}$, then P lies on $\overrightarrow{AD}$ and $P \neq A$.



(1) Given: P is on $\overrightarrow{AD}$, $P \neq A$, $\overline{PM} \perp \overleftrightarrow{AB}$, $\overline{PN} \perp \overleftrightarrow{AC}$.

To prove: P is in the interior of $\angle BAC$; $PM = PN$.

1.	P is in the interior of $\angle BAC$.	1.	P is on $\overrightarrow{AD}$, $P \neq A$, and definition of bisector of an angle.
2.	$\overline{AP} \cong \overline{AP}$.	2.	Segment is congruent to itself.
3.	$\angle PAM \cong \angle PAN$.	3.	Definition of bisector.
4.	$\angle PMA \cong \angle PNA$.	4.	Right angles are congruent.
5.	$\Delta PMA \cong \Delta PNA$.	5.	S.A.A. Theorem.
6.	$PM = PN$.	6.	Corresponding parts.



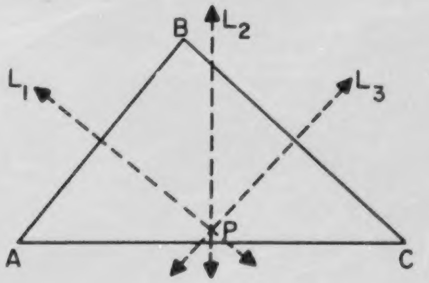
(2) Given: P is in the interior of $\angle BAC$, $\overline{PM} \perp \overleftrightarrow{AB}$, $\overline{PN} \perp \overleftrightarrow{AC}$, $PM = PN$.

To prove: $P \neq A$; P lies on $\overrightarrow{AD}$.

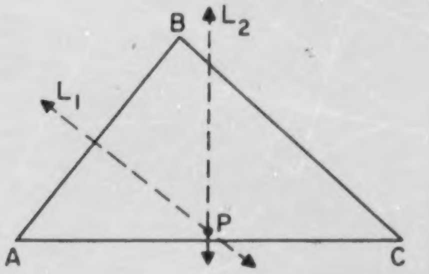
1.	$P \neq A$.	1.	Definition of interior of an angle.
2.	$\overline{PM} \cong \overline{PN}$.	2.	Definition of congruent segments.
3.	$\overline{PA} \cong \overline{PA}$.	3.	Segment is congruent to itself.
4.	$\angle PMA$ and $\angle PNA$ are right angles.	4.	Given.
5.	$\Delta PMA \cong \Delta PNA$.	5.	Hypotenuse-Leg Theorem.
6.	$\angle PAM \cong \angle PAN$.	6.	Corresponding parts.
7.	P lies on $\overrightarrow{AD}$.	7.	Definition of bisector of an angle.

As a first application of set characterization we will prove three concurrence theorems analogous to Theorem 9-27 on concurrence of medians.

Theorem 14-2. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the three vertices of the triangle.



Proof: Let L_1, L_2 and L_3 be the perpendicular bisectors of the three sides $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$. If L_1 and L_2 were parallel then $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ would be parallel. (Why?) Therefore, L_1 and L_2 intersect in a point P.



By Theorem 6-2, $AP = BP$, because P is on L_1. And $AP = CP$, because P is on L_2. Therefore $BP = CP$. By Theorem 6-2, this means that P is on L_3. Therefore P is on all three of the perpendicular bisectors and $AP = BP = CP$, which was to be proved.

Corollary 14-2-1. There is one and only one circle through three non-collinear points.

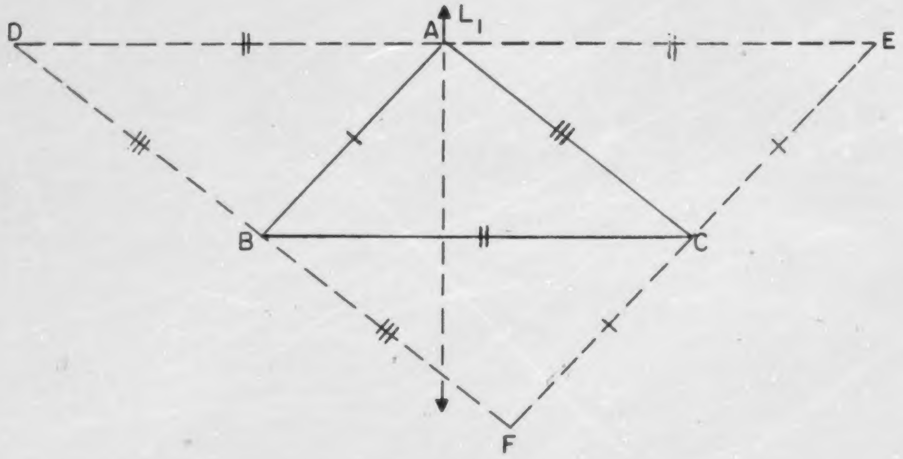
Corollary 14-2-2. Two distinct circles can intersect in at most two points.

Suggestion for proof: If two circles could intersect in three points, the three points could be either collinear or non-collinear. Use Theorem 13-2 and Corollary 14-2-1 to show that this is impossible in each case.

Theorem 14-3. The three altitudes of a triangle are concurrent.

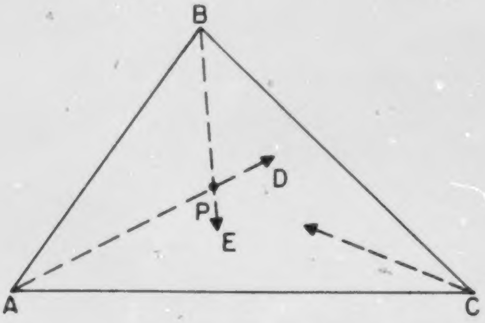
Up to now, we have been using the word altitude mainly in two senses: It means (1) the perpendicular segment from a vertex of a triangle to the line containing the opposite side or (2) the length of this perpendicular segment. In Theorem 14-3, we are using the word altitude in a third sense: It means the line that contains the perpendicular segment.

Theorem 14-3 is easy to prove - if you go about it in exactly the right way.



Given Δ ABC, we draw through each vertex a line parallel to the opposite side. These three lines determine a triangle Δ DEF. Opposite sides of a parallelogram are congruent. Therefore BC = AE and BC = DA. Therefore DA = AE. Therefore the altitude from A, in Δ ABC, is the perpendicular bisector of $\overline{DE}$. (This is L_1 in the figure.) For the same reasons, the other two altitudes of Δ ABC are the perpendicular bisectors of the sides of Δ DEF. Since the perpendicular bisectors are concurrent, so also are the three altitudes.

Theorem 14-4. The angle bisectors of a triangle are concurrent in a point equidistant from the three sides.

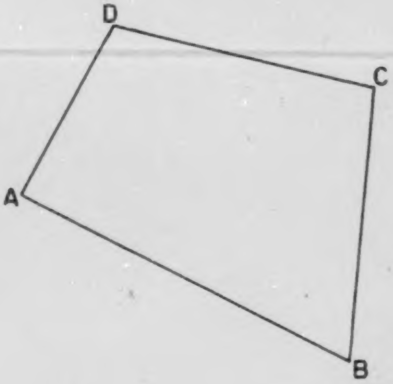


Proof: Let P be the intersection of the bisectors $\overrightarrow{AD}$ and $\overrightarrow{BE}$. By Theorem 14-1, P is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, because P is on the bisector of $\angle$ A. And P is equidistant from $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$, because P is on the bisector of $\angle$ B. Therefore P is equidistant from $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$. Therefore, by Theorem 14-1, P is on the bisector of $\angle$ C. Therefore, the three bisectors have the point P in common and P is equidistant from $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$, which was to be proved.

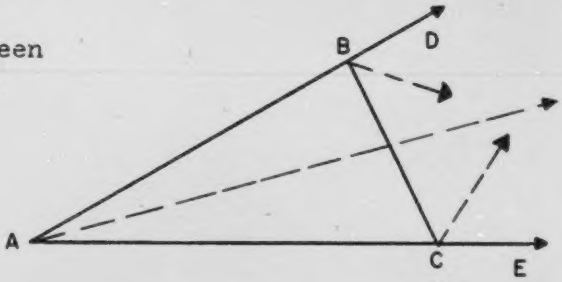
Problem Set 14-2b

1. A line intersects the sides of $\angle ABC$ in P and Q. Find a point of $\overleftrightarrow{PQ}$ which is equally distant from the sides of the angle.

2. Imagine this figure as a city park. The park commission plans to place a drinking fountain at a point which shall be equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ and also equidistant from D and C. Explain how to find this point.



3. Prove the following theorem: Given $\angle DAE$ and B, C points on $\overrightarrow{AD}$, $\overrightarrow{AE}$, between A and D and A and E respectively, then the bisectors of the angles BAC, DBC, BCE, are concurrent.



4. Given the three lines determined by the sides of a triangle, show that there are exactly four points each of which is equidistant from all three lines.

5. Mark points M and N 2 inches apart and draw circles with radii $\frac{1}{2}$ inch, 1 inch, 2 inches and 3 inches using both M and N as centers each time.
   Note that some of the circles with center at M intersect circles with center at N, but that there are two kinds of situations in which they do not. Describe these two situations.

6. Sketch several different quadrilaterals, and in each sketch the bisectors of each of the four angles. From your sketches does it appear that these angle bisectors are always concurrent? Can you think of any special type of quadrilateral whose angle bisectors are concurrent? Can you think of a general way of describing those quadrilaterals whose angle bisectors are concurrent? (Hint: If the angle bisectors are concurrent, the point of concurrency is equidistant from all four sides.)

7. A quadrilateral is cyclic if its four vertices lie on a circle. Prove that the perpendicular bisectors of the four sides and the two diagonals of a cyclic quadrilateral are concurrent.

8. What is the set of points which are the vertices of right triangles having a given segment $\overline{AB}$ as hypotenuse?

14-3. Intersection of Sets.

Consider the following problem: In a given plane E how many points are there which are at a given distance r from a given point A of E and which are also equidistant from two given points B and C of E?

Such a point P is required to satisfy two conditions;

(1) $AP = r$, (2) $BP = CP$.

Consider these conditions one at a time. If P satisfies (1) then P can be anywhere on the circle with center A and radius r. In other words, the set of points satisfying (1) is this circle.

Similarly, by Theorem 6-2, the set of points satisfying (2) is a line, the perpendicular bisector of $\overline{BC}$. If P is to satisfy both conditions it must lie on both sets; that is, P must be a point of the intersection of the two sets. Since the intersection of a line and a circle can be two points, one point, or no points, the answer to our problem is two, one, or none, depending on the relative positions of A, B and C and the value of r. The method illustrated here is a very useful one, since it enables us to consider a complicated problem a piece at a time and then put the pieces together as a final step. If you refer to the proofs of Theorems 14-2 and 14-4 you will see that this was the basic method of the proof. In Theorem 14-2, for example, we found the point P as the intersection of the set L_1 defined by $PA = PB$ and the set L_2 defined by $PA = PC$.

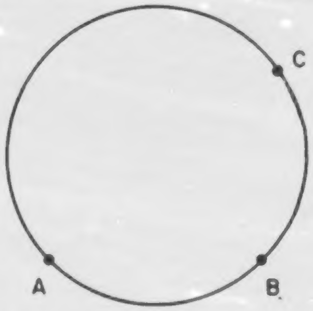
Most of the constructions which are to be discussed in the next sections are based on the method of intersection of sets.

Problem Set 14-3

1. $\overline{AB}$ is a segment 6 inches long in a plane E. Describe the location of points P in E, 4 inches from A, and 5 inches from B.

2. $\overline{AB}$ is a segment 4 inches long in a plane E. C and D are points of E such that D is on $\overline{AB}$, $\overline{CD} \perp \overline{AB}$ and $\overline{CD}$ is 3 inches long. Describe the set of points P which are equidistant from A and B, and 5 inches from C.

*3. On a circular lake there are three docks, A, B, C. Draw a diagram indicating those points on the lake which are closer to A than to B or C.



4. Are there any points in a plane that satisfy the following conditions? If there are, tell how many such points and how each is determined. Make a sketch to illustrate your answer.

   Given $BC = 6$ inches.

   a. 4 inches from B and 3 inches from C.

   b. 10 inches from B and 10 inches from C.

   c. 10 inches from B and equidistant from C and B.

   d. 2 inches from B and 4 inches from C.

14-4. Constructions with Straight-edge and Compass.

A practical problem of some importance is that of drawing a figure with accuracy. This is the job of a draftsman, and he uses many instruments to facilitate his work, such as rulers, compasses, dividers, triangles, T-squares, and a host of other devices.

The corresponding geometric process is generally called "constructing" rather than "drawing", but the idea is the same. We allow ourselves the use of certain instruments, and the basic problem is to show how, with these instruments, we can construct various figures.

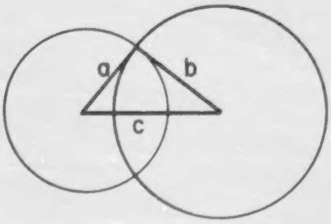
Of course our constructions will depend on the instruments we use. Thus far in our text we have been considering the ruler and the protractor as our fundamental instruments, although we would have had to introduce a compass in Chapter 13 to construct circles. Various other combinations of instruments have been considered, but the most interesting is still the combination used by the ancient Greeks, the straight-edge and compass. We shall devote the rest of this chapter to constructions with these instruments.

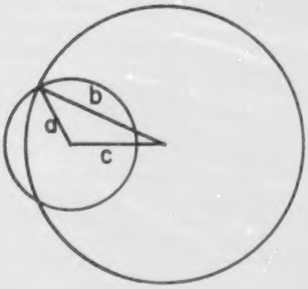
A straight-edge is simply a device to draw lines. It has no marks on its edge and so we cannot measure distances with it. With a compass we can draw a circle with a given center and a given radius. We have no means of measuring angles.

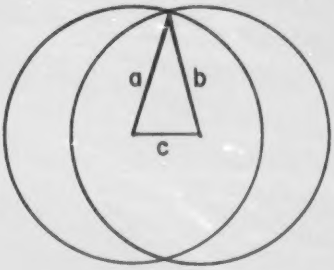
Most of our constructions will depend on the intersection properties of two lines, of a line and a circle, or of two circles. The first of these three cases has been considered in such places as Theorem 3-1, the Plane Separation Postulate and the Parallel Postulate. The case of a line and circle was taken care of by Theorem 13-2. But we still have the case of two circles to consider. As might be expected this is the most complicated of the lot, both to state and to prove. In fact, the proof is so complicated that we do not give it here at all, but put it in Appendix IX. Here is the theorem:

Theorem 14-5. (The Two Circle Theorem.) If two circles have radii a and b, and if c is the distance between their centers, then the circles intersect in two points, one on each side of the line of centers, provided each one of a, b, c is less than the sum of the other two.

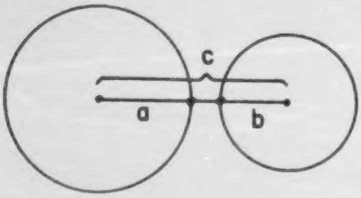
Some of the situations in which the inequalities stated in the theorem are all satisfied and the circles intersect are illustrated below:



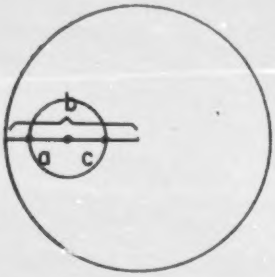




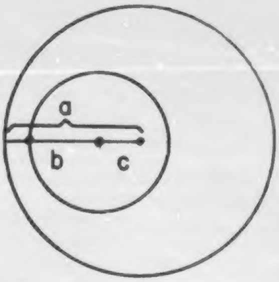
That the inequality condition imposed on a, b, c is important is shown by these cases in which one of the inequalities stated in the theorem is not satisfied and the circles do not intersect:



$c > a + b$



$b > a + c$



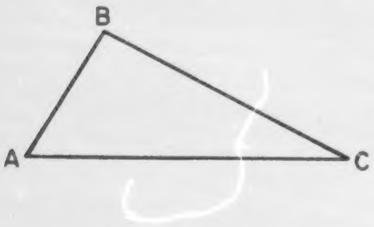
$a > b + c.$

14-5. Elementary Constructions.

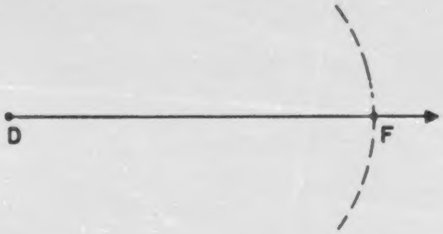
In this section we show how to do various simple constructions which will be needed as steps in the more difficult ones. All these constructions will be in a given plane. Constructions will be numbered in the same way as theorems.

Construction 14-6. To copy a given triangle.

Suppose we have given Δ ABC. We want to construct a triangle Δ DEF, congruent to Δ ABC, with the side $\overline{DF}$ lying on a given ray with D as end-point.

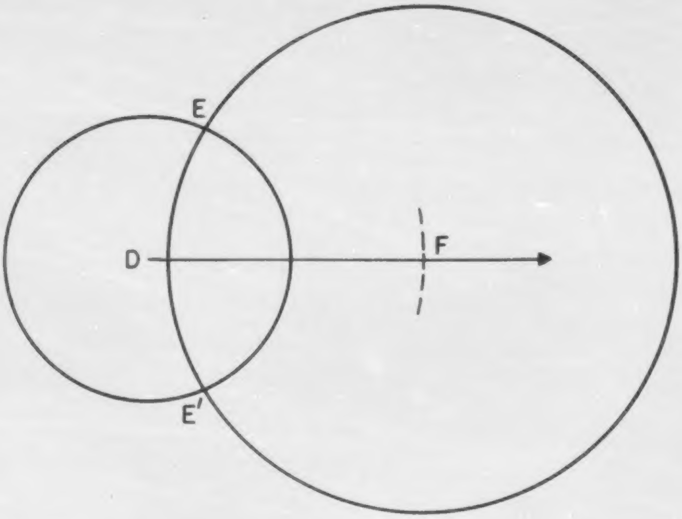


Step 1. With the compass, construct a circle with center at D and radius AC. This intersects the given ray in a point F, and DF = AC. In the figure, we show only a short arc of the circle.



Step 2. With the compass, construct a circle with center at D and radius AB.

Step 3. Construct a circle with center at F and radius BC.

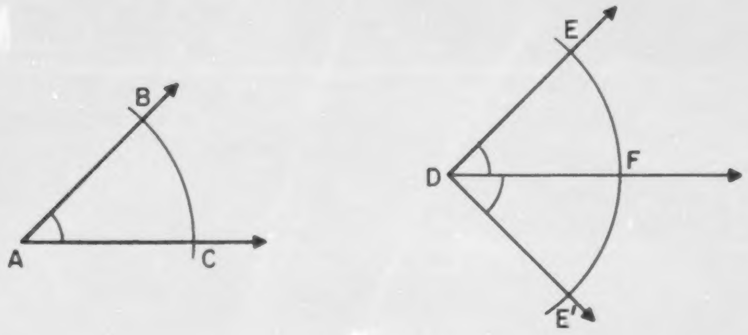


These two circles seem to intersect; and by the Two Circle Theorem they must intersect, because each of the numbers AC, AB, and BC is less than the sum of the other two, by Theorem 7-7.

Either of the points E, E' will do as the third vertex of our triangle. We draw the sides with our straight-edge, and we know by the S.S.S. Theorem that $\Delta DEF \cong \Delta ABC$.

You may remember that in proving the S.S.S. Theorem we had the problem of copying a triangle. It is worth while to review the old method and compare it with the new one. (In the proof of the S.S.S. Theorem we copied the triangle with ruler and protractor, using the S.A.S. Postulate to verify that the construction really worked.)

Construction 14-7. To copy a given angle.

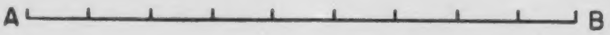


Here we have given an angle with vertex at A, and we have given a ray with end-point at D. We want to construct the two angles, having the given ray as a side, congruent to the given angle.

With A as center, we construct an arc of a circle intersecting the sides of the angle in points B and C. With D as center construct a sufficiently large arc of a circle of the same radius, intersecting the given ray in F. With F as center and BC as radius construct arcs of a circle, intersecting the circle with center D in E and E'. Construct ray $\overrightarrow{DE}$ and ray $\overrightarrow{DE'}$. By S.S.S. Theorem $\Delta DEF \cong \Delta ABC$, and hence $\angle EDF \cong \angle BAC$. Similarly, $\angle E'DF \cong \angle BAC$.

Problem Set 14-5a

1. For your convenience, we give $\overline{AB}$ 9 cm. long.



Construct a triangle with sides of the following lengths:

a. 5 cm., 6 cm., 8 cm.

b. 7 cm., 5 cm., 3 cm.

c. 3 cm., 3 cm., 3 cm.

d. 4 cm., 7 cm., 3 cm.

2. Make a triangle ABC on your paper and construct $\triangle AB'C$ congruent to $\triangle ABC$ using $\overline{AC}$ as a side in each and the A.S.A. Theorem as your method.

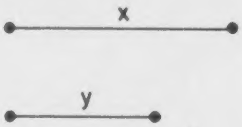
3. Draw on your paper a triangle ABC and a segment $\overline{MH}$ about twice as long as $\overline{AB}$. With M as vertex construct $\angle HMQ \cong \angle A$. With H as vertex construct $\angle QHM \cong \angle B$. $\angle Q \cong$ _____. $\frac{AB}{MH} =$ _____ $=$ _____.

4. a. Prove that it is always possible to construct an equilateral triangle having a given segment as one of its sides.

   b. Under what conditions is it possible to construct an isosceles triangle having one given segment as its side and another given segment as its base?

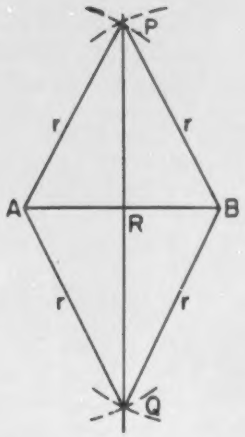
5. a. Construct an equilateral triangle with x as the length of one side.

   b. Construct an isosceles triangle with y as the length of the base and x as the length of one of the congruent sides.



Construction 14-8. To construct the perpendicular bisector of a given segment.

Given a segment $\overline{AB}$.



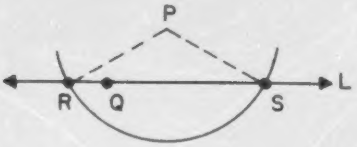
Step 1. Using an appropriate radius r, construct a circle with center at A and a circle with center at B. If r is chosen in a suitable way, these two circles will intersect in two points P and Q, lying on opposite sides of $\overleftrightarrow{AB}$.

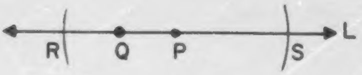
(Question: What condition should r satisfy, to ensure that the circles will intersect in this way? Can you think of a particular value of r that is sure to work? Of course, only one value of r is needed for the construction.)

Step 2. Construct the line $\overleftrightarrow{PQ}$, intersecting $\overleftrightarrow{AB}$ at R. We need to show that this line is the perpendicular bisector of $\overline{AB}$. By Theorem 6-2, R and S, being each equidistant from A and B lie on the perpendicular bisector of $\overline{AB}$. Since two points determine a line, $\overleftrightarrow{PQ}$ *is* the perpendicular bisector.

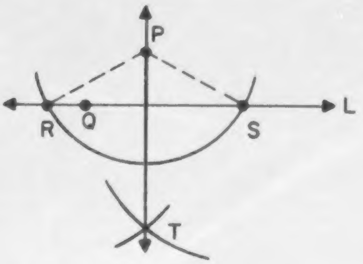
Corollary 14-8-1. To bisect a given segment.

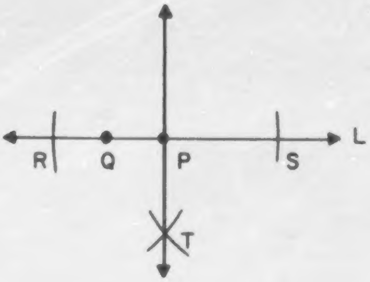
Construction 14-9. To construct a perpendicular to a given line through a given point.





Step 1. Given P and L. Let Q be any point of L. Draw a circle with center P and radius r, where r is greater than PQ. L then contains a point of the interior of the circle (namely, Q) and by Corollary 13-2-6 intersects the circle in two points R and S.



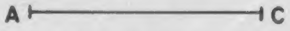


Step 2. With R as center and radius greater than $\frac{1}{2}$ RS construct a suitable arc of a circle. With S as a center and the same radius, construct an arc of a circle intersecting this in T. Then, as in Construction 14-8, P and T are each equidistant from R and S, and hence, $\overleftrightarrow{PT} \perp \overleftrightarrow{RS}$.

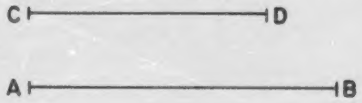
Problem Set 14-5b

1. Construct an isosceles right triangle.

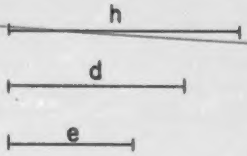
2. Construct a square in which a diagonal is congruent to $\overline{AC}$.



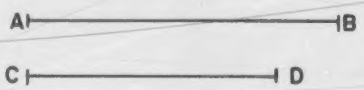
3. Construct a rhombus whose diagonals are congruent to $\overline{AB}$ and $\overline{CD}$.



4. Construct a triangle given any altitude h and the segments d and e of the side it intersects.

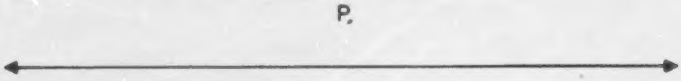


5. Construct a parallelogram whose diagonals are congruent to $\overline{AB}$ and $\overline{CD}$ and which determine a 60° angle.

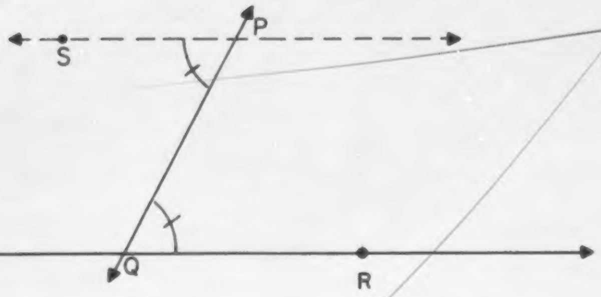


6. Construct a segment whose length is the geometric mean of $\overline{AB}$ and $\overline{CD}$ in Problem 5. (Hint: Refer to Problem 10 of Problem Set 13-4a.)

Construction 14-10. To construct a parallel to a given line, through a given external point.

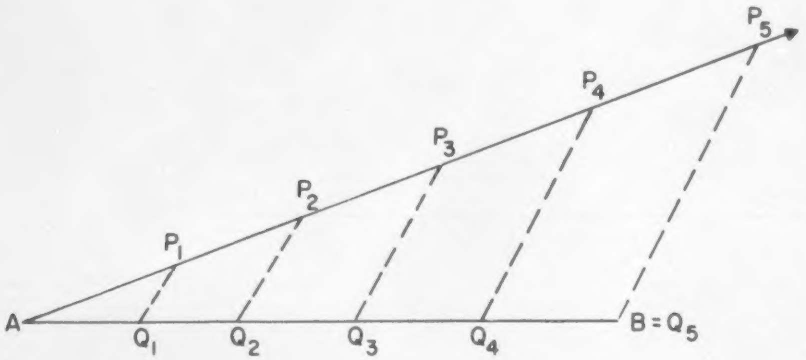


Step 1. Take any point Q of the line, and join P to Q by a line.



Step 2. Now construct $\angle QPS$, congruent to $\angle PQR$, with S and R on opposite sides of $\overleftrightarrow{PQ}$. Step 2 is an example of Construction 14-7. Then $\overleftrightarrow{PS}$ is parallel to $\overleftrightarrow{QR}$, as desired.

Construction 14-11. To divide a segment into a given number of congruent segments.



Given $\overline{AB}$, we want to divide $\overline{AB}$ into n congruent segments. (In the figure, we show the case $n = 5$.)

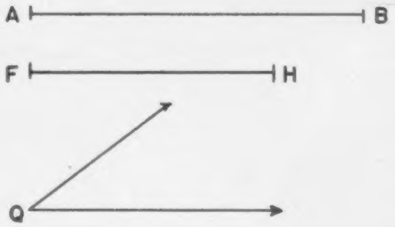
Step 1. Draw any ray starting at A, not on the line $\overleftrightarrow{AB}$. Starting at A, lay off n congruent segments $\overline{AP_1}$, $\overline{P_1P_2}$, ..., $\overline{P_{n-1}P_n}$, end to end, on the ray. (The length does not matter, as long as they have the same length; we simply choose P_1 at random, and then use the compass to lay off $\overline{P_1P_2} \cong \overline{AP_1}$, and so on.)

Step 2. Join P_n to B by a line. Through the other points P_1, P_2, ..., P_{n-1} construct lines parallel to $\overleftrightarrow{P_nB}$. (This can be done; it is Construction 14-10.)

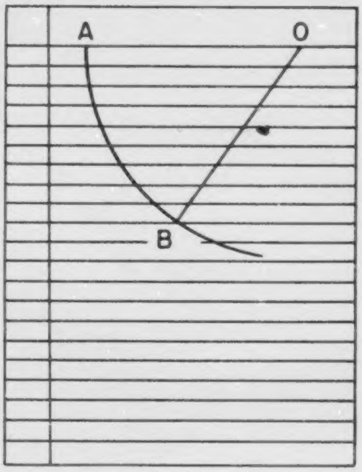
These lines intersect $\overleftrightarrow{AB}$ in points Q_1, Q_2, ..., Q_{n-1}. The points Q_1, Q_2, ..., Q_{n-1} divide $\overline{AB}$ into n congruent segments. (See Corollary 9-26-1.)

Problem Set 14-5c

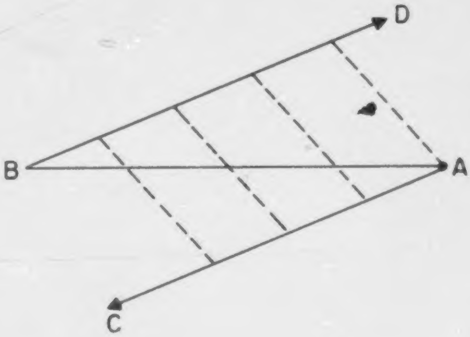
1. Construct a parallelogram with two sides and included angle congruent to $\overline{AB}$, $\overline{FH}$, and $\angle Q$.



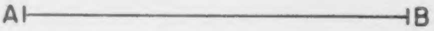
2. This drawing shows how Bob Langford used a sheet of ruled paper to divide a segment $\overline{AO}$ in 9 parts of equal length. Explain how he could have divided it into other numbers of congruent parts. (Assume that the lines of paper are evenly spaced.)



3. This figure illustrates still another method for dividing a segment into any number of congruent parts. Here $\overleftrightarrow{AC}$ is any convenient line, and $\overleftrightarrow{BD}$ is drawn parallel to $\overleftrightarrow{AC}$. The same number of congruent segments is marked off on each, and the corresponding points are joined. Prove that the method is correct.

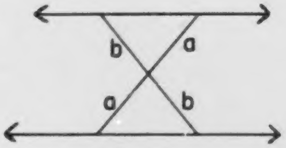


4. If the length of $\overline{AB}$ is the perimeter of an equilateral triangle, construct the triangle.

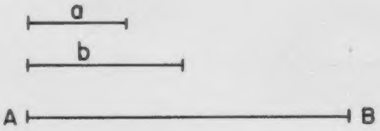


5. Given $\overline{AB}$, construct an isosceles triangle in which AB is the perimeter and in which the length of one of the congruent sides is twice the length of the base.

6. This figure illustrates another method of making one line parallel to another which is useful in outdoor work. Explain the method and show that it is correct.



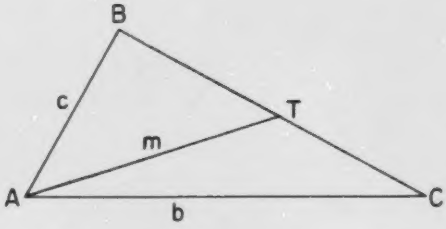
7. Divide a given line segment $\overline{AB}$ into two segments whose ratio is that of two given segments of lengths a and b. (Hint: Use a construction similar to that of Construction 14-11.)



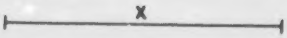
*8. Construct a triangle ABC, given the lengths of $\overline{AB}$, $\overline{AC}$, and the median from A to $\overline{BC}$.

Given: Lengths c, b, m.

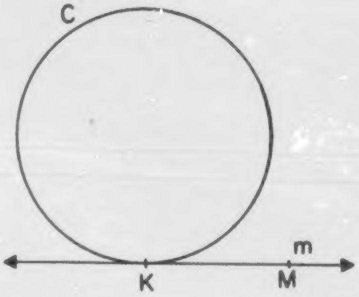
To construct: ΔABC so that $AB = c$, $AC = b$, median $AT = m$.

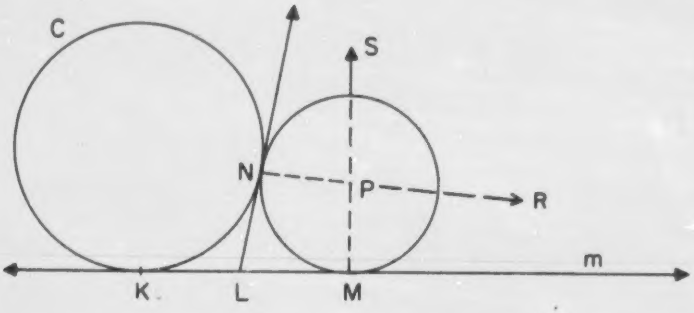


*9. Given x as the median to one of the congruent sides of an isosceles triangle in which the medians to these sides are perpendicular to each other. Construct the triangle.



*10. Given a circle C tangent to a line m at K. Construct a circle tangent to C and also tangent to m at a given point M. (Hint: Analyze the diagram below in which P is the center of the required circle, N the point of tangency and $\overleftrightarrow{LN}$ the common tangent at N.)

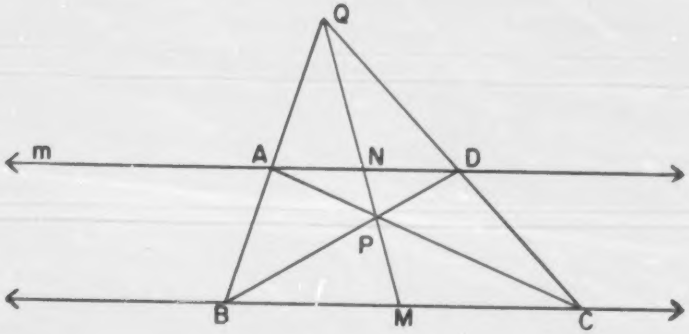




*11. Construct a common external tangent to two given circles.

*12. Given a triangle ABC in which each angle has measure less than 120, construct a point P in the plane of the triangle such that $m\angle APB = m\angle BPC = m\angle APC$.

*13. The figure shows how a segment can be bisected using a line parallel to it, by means of a straight-edge only. That is, given line $m \parallel \overline{BC}$, take Q as any point not on $\overleftrightarrow{BC}$ or m, and draw $\overleftrightarrow{QB}$ and $\overleftrightarrow{QC}$ meeting m at A and D. Then draw $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$, which meet at P. Then $\overleftrightarrow{QP}$ bisects $\overleftrightarrow{BC}$ at M. Prove this.

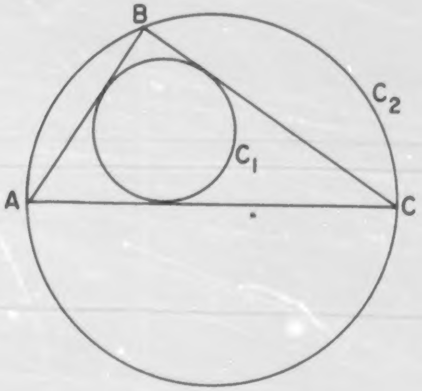


(Hint: The proof will include these three proportions: $\frac{MB}{MC} = \frac{ND}{NA}$, $\frac{MB}{NA} = \frac{MC}{ND}$ and $\frac{MB}{MC} = \frac{MC}{MB}$.)

*14. Given two parallel lines m and n, at a distance d from each other, find the set of all points P such that the distance from P to m is k times the distance from P to n, where k is a given positive number.

14-6. Inscribed and Circumscribed Circles.

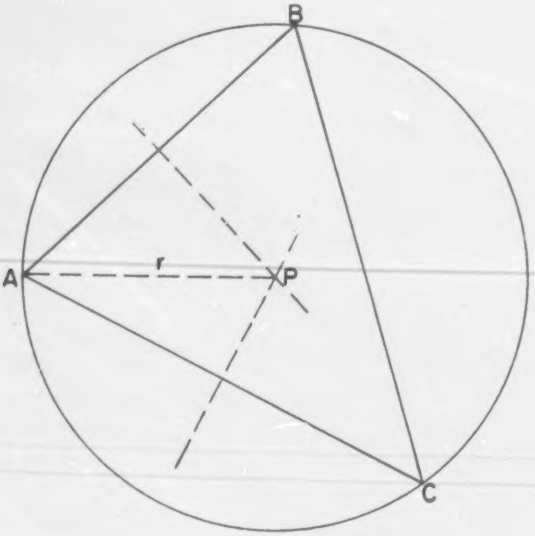
Definitions: A circle is inscribed in a triangle, or the triangle is circumscribed about the circle, if each side of the triangle is tangent to the circle. A circle is circumscribed about a triangle, or the triangle is inscribed in the circle if each vertex of the triangle lies on the circle.



In this figure Δ ABC is inscribed in C_2 and circumscribed about C_1. C_1 is inscribed in Δ ABC and C_2 is circumscribed about Δ ABC.

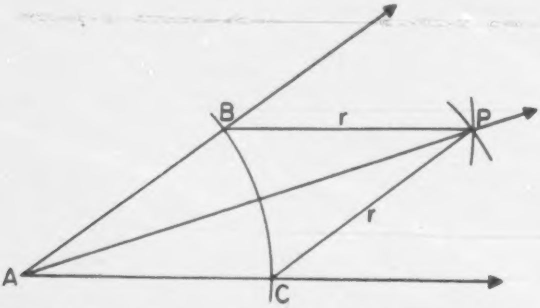
In this section we will learn how to construct with straightedge and compass the inscribed circle and the circumscribed circle, for any triangle.

Construction 14-12. To circumscribe a circle about a given triangle.



Step 1. Construct the perpendicular bisectors of two sides of the triangle. This can be done by two applications of Construction 14-8. The two lines meet at a point P. By Theorem 14-2, P also lies on the perpendicular bisector of the third side. By Theorem 6-2, this means that P is equidistant from the three vertices A, B, and C, that is, AP = BP = CP. Construct the circle with center at P, passing through A. Then the circle also passes through B and C.

Construction 14-13. To bisect a given angle.

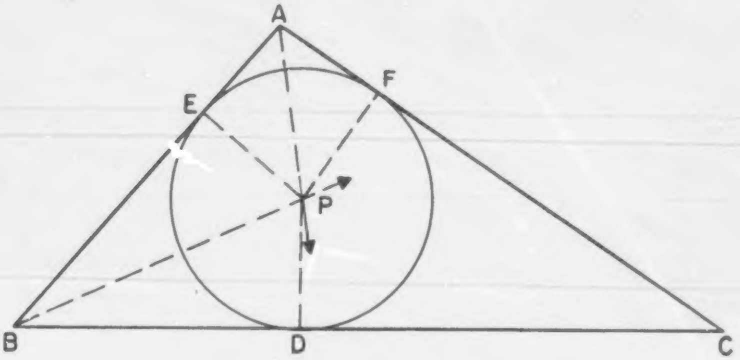


Step 1. Construct any circle with center at A, intersecting the sides of the given angle in points B and C. Then AB = AC.

Step 2. Construct circles with centers at B and at C, and with the same radius r, where $r > \frac{1}{2} BC$. By the Two-Circle Theorem these circles intersect in two points, one on each side of $\overleftrightarrow{BC}$. Let P be the point on the side opposite to A.

Step 3. Construct the ray $\overrightarrow{AP}$. By the S.S.S. Theorem, $\triangle BAP \cong \triangle CAP$. Therefore $\angle BAP \cong \angle CAP$, as desired.

Construction 14-14. To inscribe a circle in a given triangle.



Step 1. Bisect $\angle A$ and $\angle B$, and let P be the point where the bisectors intersect. By Theorem 14-4, P also lies on the bisector of $\angle C$.

Step 2. Construct a perpendicular $\overline{PD}$, from P to $\overline{BC}$. Construct a circle with center at P, passing through D. We need to show that the circle is tangent to all three sides of $\triangle ABC$.

(1) The circle is tangent to $\overline{BC}$, because $\overline{BC}$ is perpendicular to the radius $\overline{PD}$. (See Corollary 13-2-2.)

(2) By Theorem 14-1, P is equidistant from $\overline{AB}$ and $\overline{BC}$. Therefore the circle contains the point E which is the foot of the perpendicular from P to $\overline{AB}$. Therefore the circle is tangent to $\overline{AB}$.

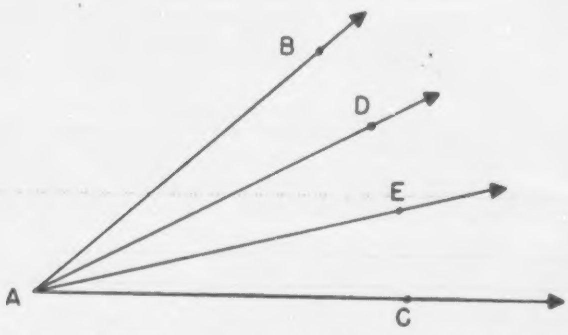
The proof of tangency for the third side is exactly the same.

Notice that if all you want is a fairly convincing drawing you can merely construct the two bisectors, put the point of the compass at P, and then adjust the compass so that its pencil-point will barely reach $\overline{BC}$. You have to drop the perpendicular $\overline{PD}$, however, to get a construction which is theoretically exact.

14-7. The Impossible Construction Problems of Antiquity.

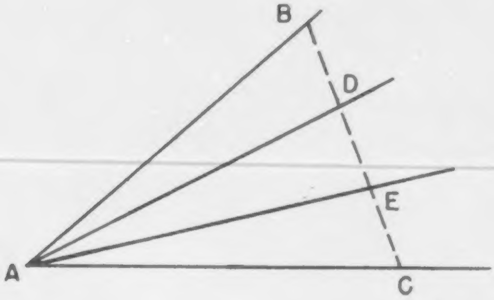
The ancient Greeks discovered all of the straight-edge-and-compass constructions that you have studied so far, together with a large number of more difficult ones. There were some construction problems, however, which they tried long and hard to solve, with no success whatever.

(1) The angle-trisection problem.



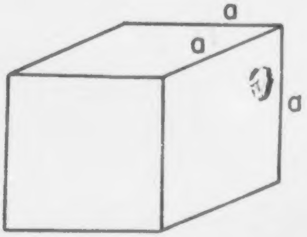
Given an angle $\angle BAC$, we want to construct two rays $\overrightarrow{AD}$ and $\overrightarrow{AE}$ (with points D and E in the interior of $\angle BAC$) which trisect $\angle BAC$. That is, we want $\angle BAD \cong \angle DAE \cong \angle EAC$.

Nobody has found a way to do this with straight-edge and compass. The first thing that most people try is to take AB = AC, draw $\overline{BC}$, and then trisect $\overline{BC}$ with points D and E.



But this doesn't work; in fact, nothing has been found that works.

(2) The duplication of the cube. A cube of edge a has volume a^3.



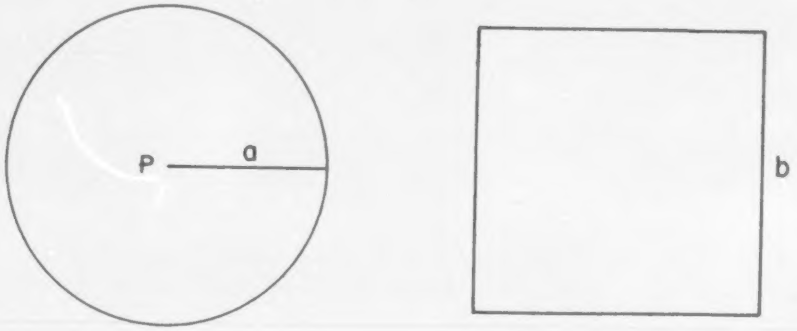
$V = a^3$.

Suppose we have given a segment of length a. We want to construct a segment of length b, such that a cube of edge b has exactly twice the volume of a cube of edge a. (Algebraically, of course, this means that $b^3 = 2a^3$, or $\frac{b}{a} = \sqrt[3]{2}$.

This problem was attacked, over a long period, by the best mathematicians in Greece, who were very brilliant men indeed, but none of them had any success with this problem.

There is a curious myth in connection with this problem. A plague threatened the population of a certain Greek town, and the inhabitants consulted the oracle at Delphi to find out which god was angry and why. The answer they got from the oracle was that Apollo was angry. There was an altar to Apollo, in the town, consisting of a cube of solid gold, and Apollo wanted his altar to be exactly twice as big. The people went home from Delphi and built a new altar, twice as long along an edge as the old one. The plague then got worse instead of better. The people thought again, and realized that the new altar was eight times as big as the old one, that is, it had eight times as much volume. This raised the problem of the duplication of the cube, but the local mathematicians were unable to solve the problem. Thus the first attempt to apply mathematics to public health was a total failure.

(3) Squaring the circle. Suppose we have given a circle. We want to construct a square whose area is exactly the same as that of the circle.



$A = \pi a^2$

$A = b^2$

Algebraically, this means that $b = a\sqrt{\pi}$.

These three problems occupied many people for more than two thousand years. Various attempts were made to solve them with straight-edge and compass constructions. Finally it was discovered, in modern times, that all three of these problems are impossible. Impossibility in mathematics does not mean the same thing as "impossibility" in every day life, and so it calls for some explanation.

Ordinarily, when we say that something is "impossible," we mean merely that it is extremely difficult, or that we don't happen to see how it can be done, or that nobody has found a way to do it -- so far. Thus people used to say that it was "impossible" to build a flying machine, and people didn't stop this until the first airplane was built. It is supposed to be "impossible" to find a needle in a haystack, and so on.

Mathematical impossibility is not like this. In mathematics, there are some things that really can't be done, and it is possible to *prove* that they can't be done.

(1) A very simple example is this: No matter how clever and persistent you may be, you can't find a whole number between 2 and 3, because there isn't any such whole number.

(2) If the above example seems too trivial to take seriously, consider the following situation. We start with the integers, positive, negative and 0. We are allowed to perform additions, subtractions, multiplications, and divisions. A number is called *constructible* if we can get to it, starting from the integers, by a finite number of such steps. For example, the following number is *constructible*:

$$\frac{\frac{5}{2}-\frac{17}{37}}{\frac{3}{4}+\frac{7}{3}}+\frac{\frac{1}{7}+\frac{3}{5}}{\frac{9}{10}-\frac{37}{47}}$$

To get to it requires 15 steps.

Now suppose that the problem before us is to construct the number $\sqrt[3]{2}$. This problem is impossible of solution just as the old Greek problems are. The point is, the numbers that can be constructed according to the rules that we have agreed to are all rational numbers. And $\sqrt[3]{2}$ just isn't this kind of a number. There is no use in hunting for it among the constructible numbers, because that isn't where it is.

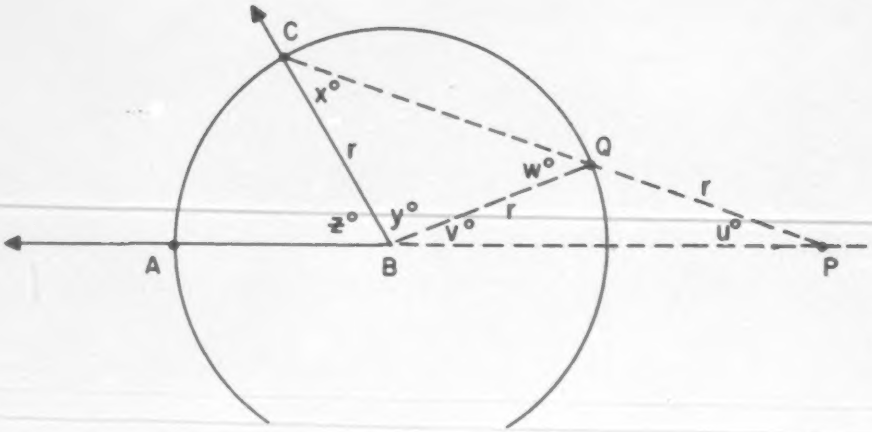
Problems of constructibility with straight-edge and compass are closely analogous to this second illustration. Starting with the integers, there are certain numbers that we can "construct" by elementary arithmetic, but these numbers do not happen to include $\sqrt[3]{2}$.

Starting with a segment, $\overline{AB}$, there are certain figures that we can construct with straight-edge and compass, but these figures do not happen to include any segment $\overline{CD}$ for which $CD^3 = 2 \cdot AB^3$. This is what we mean when we say that the duplication of the cube with straight-edge and compass is impossible of solution.

The angle-trisection problem deserves some further discussion.

(1) Some angles can be trisected with straight-edge and compass. For example, a right angle can be so trisected. When we say that the angle-trisection problem is impossible of solution, we mean that there are some angles for which no trisecting rays can be constructed.

(2) The angle-trisection problem becomes solvable if we change the construction rules very slightly, by allowing ourselves to make two marks on the straight-edge. Once the two marks are made, we proceed as follows:



Given an angle with vertex B, we draw a circle with center at B and radius r equal to the distance between the two marks on the straight-edge. The circle intersects the sides of the given angle in points A and C. We want to construct an angle whose measure is $\frac{1}{3}(m\angle ABC)$.

Place the straight-edge so that (1) it passes through C. Now manipulate the straight-edge by sliding and rotating it about C so that (2) one marked point Q lies on the circle and (3) the other marked point P lies on the ray opposite to $\overrightarrow{BA}$. We will show that $m\angle BPC = \frac{1}{3}(m\angle ABC)$. In terms of the angle-measures indicated in the figure, the main steps in the proof are as follows; you should find the reasons in each case:

(1) $v = u$.

(2) $w = u + v = 2u$.

(3) $x = w = 2u$.

(4) $z = x + u = 3u$.

Equation (4) is, of course, what we wanted to prove. Once we have $\angle BPC$, it is easy to draw the trisecting rays in the interior of $\angle ABC$, by two applications of Construction 14-7.

Problem Set 14-7

1. Find the set of points which are the intersections of the bisectors of the base angles of parallelograms that have a fixed segment as base.

2. Explain how to construct an angle of

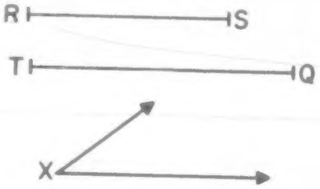
   a. 45°;
   b. 30°;
   c. $22\frac{1}{2}^\circ$;
   d. 135°;
   e. 120°;
   f. 75°;
   g. 105°;
   h. $67\frac{1}{2}^\circ$.

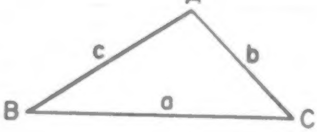
   Mention three other angles you could construct.

3. In dealing with triangles it is helpful to be able to designate the parts by brief symbols. A notation frequently used is as follows:

   A, B, C, for the three vertices;
   a, b, c, for the lengths of the opposite sides;
   h_a, h_b, h_c for the altitudes to $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$;
   t_A, t_B, t_C for bisectors of angles A, B, C;
   m_a, m_b, m_c for medians to sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$.

In each of the following problems, we wish to construct a triangle satisfying certain conditions. For example, we might give two segments $\overline{RS}$ and $\overline{TQ}$ and an angle, say $\angle X$, and require that a triangle ABC be constructed so that $\overline{AB} \cong \overline{RS}$, $\overline{BC} \cong \overline{TQ}$, and $\angle B \cong \angle X$.





For brevity, we shall state such a problem in the form, "Construct a triangle given two sides and the included angle" or "Construct Δ ABC, given c, a, and $\angle$ B." The student should do several problems of this type, rephrasing them in the more exact language used above, until he is sure that he understands the meaning of the shorter statement.

Construct Δ ABC having given:

a. a, m_a, $\angle B$.

b. a, b, and $\angle X$ such that $m\angle A + m\angle B = m\angle X$.

c. a, b, h_b.

d. c, $\angle A$, t_A.

e. m_a, h_a, $\angle B$.

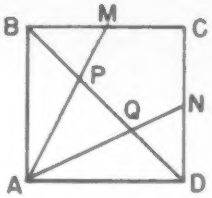
f. h_a, $\angle B$, $\angle C$.

g. $\angle C$, h_c, t_c.

h. $\angle A$, b, t_c.

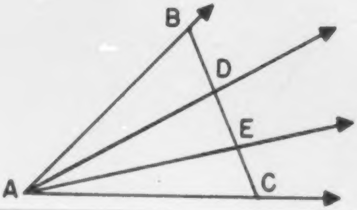
(Suggestion: Each time begin by sketching a figure showing the relationship of the given parts to help you in your analysis of the problem.)

4. Given a square ABCD with M and N the mid-points of $\overline{BC}$ and $\overline{CD}$. If $\overline{AM}$ and $\overline{AN}$ meet the diagonal $\overline{BD}$ at P and Q, prove that P and Q trisect $\overline{BD}$, but that $m\angle BAM \neq \frac{1}{3} \cdot 90$.

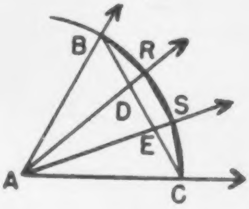


5. Show that the angle-trisection method mentioned in the text on page 504 never works, by using one of the following methods:

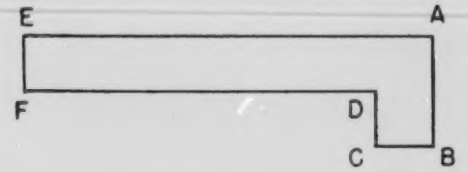
a. Suppose that for some angle it did work. Then in the diagram, $\overline{AD}$ is both the angle-bisector and the median from A in Δ BAE. The triangle is then isosceles and AB = AE (Why?). But AB = AC by construction, so the circle with center at A and radius $\overline{AB}$ is intersected by the line $\overleftrightarrow{BC}$ in three points. This is impossible.



b. Suppose it did work. Then in the diagram, let the circle with center A and radius AB intersect rays $\overrightarrow{AD}$ and $\overrightarrow{AE}$ in R and S. Then D and E will be inside the circle. (Why?) Now $\overline{RS} \parallel \overline{BC}$. (Consider the bisector of $\angle$ RAS.) Also RS > DE (Why?) Triangles ABD, ADE, and AEC all have the same area. (Why?) Now compare the areas of BDR, DRSE, and SEC to arrive at a contradiction.

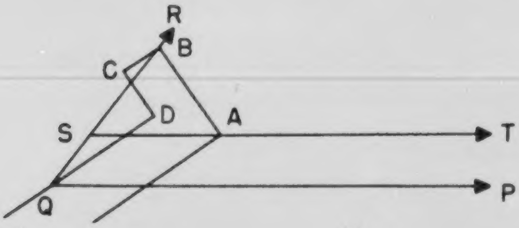


6. We hereby define a geometer's square as an instrument, made of a flat piece of cardboard or similar material, of the following shape.



The angles are all right angles and EF = CD = $\frac{1}{2}$ AB.

To trisect angle PQR with a geometer's square one first uses the long side to

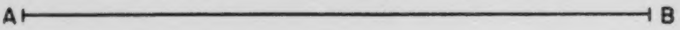


construct $\overrightarrow{ST} \parallel \overrightarrow{QP}$ at distance EF. Then place the geometer's square so that $\overleftrightarrow{DF}$ passes through Q, A lies on $\overrightarrow{ST}$, and B lies on $\overrightarrow{QR}$. Then $m\angle PQA = \frac{1}{3}(m\angle PQR)$. Prove this.

Review Problems

1. For what integral values of x is there a triangle whose sides have lengths 4, 6, x?

2. Construct a rhombus in which the perimeter has a given length AB and one angle has measure 45.



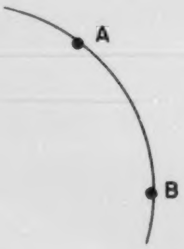
3. a. Given $\overline{AB}$, construct the set of points P in the plane such that $m\angle APB = 90$.

   b. Prove that the set you have constructed fulfills the conditions.

4. Given line L and point P in plane E. Describe the set of points in E which are a given distance d from L and a given distance r from P.

5. Sketch several quadrilaterals and, in each, sketch the perpendicular bisectors of the four sides. In general, you will find that these do not appear to be concurrent. If you can think of any special quadrilaterals whose perpendicular bisectors are concurrent, list them. Think of some general way of describing the set of quadrilaterals with this property.

6. By construction find the center of the circle of which $\widehat{AB}$ is an arc.



7. Given a segment representing the difference between the diagonal and side of a square. Construct the square.

8. Let A be the center of a circle of radius a, and B the center of a circle of radius b. If $a + b > AB$, do circles A and B always intersect?

9. ABCD is a parallelogram in a plane E. P is a point of E which is equidistant from A, B, C, and D. Prove that the parallelogram is a rectangle.

10. ABCD is a trapezoid with $\overline{AB} \parallel \overline{CD}$. Under what circumstances will there be a point P, in the plane of the trapezoid, equidistant from A, B, C, D? Can there ever be more than one such point?

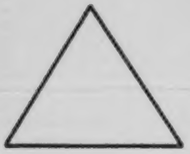
*11. Given two parallel lines ℓ and m and a transversal n, are there any points which are equidistant from ℓ, m and n? Prove that your answer is correct.

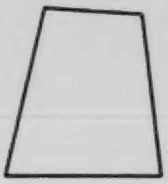
Chapter 15

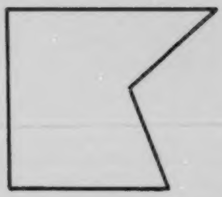
AREAS OF CIRCLES AND SECTORS

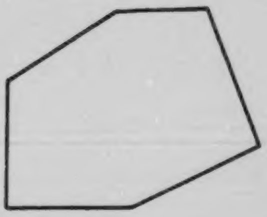
15-1. Polygons.

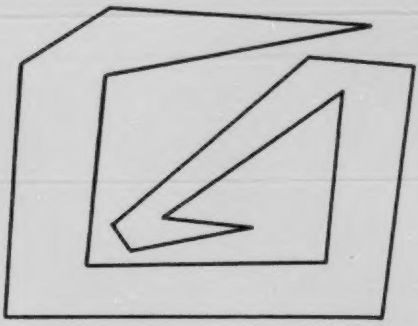
A polygon is a figure like this:



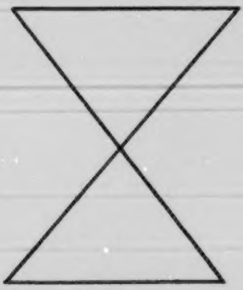








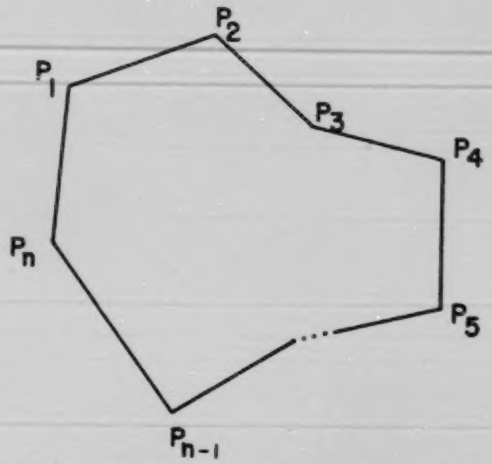
But not like this:



The idea of a polygon can be defined more precisely as follows: Suppose that we have given a sequence

$$P_1, \ P_2, \ \ldots, \ P_n$$

of distinct points in a plane. We join each point to the next one by a segment, and finally we join P_n to P_1.



In the figure, the dots indicate other possible points and segments; because we don't know how large n is. Notice that the point just before P_n is P_{n-1}, as it should be.

Definitions: Let P_1, P_2, P_3, ..., P_{n-1}, P_n be n distinct points in a plane ($n \geq 3$). Let the n segments $\overline{P_1P_2}$, $\overline{P_2,P_3}$, ..., $\overline{P_{n-1}P_n}$, $\overline{P_nP_1}$ have the properties:

(1) No two segments intersect except at their end-points, as specified;

(2) No two segments with a common end-point are collinear.

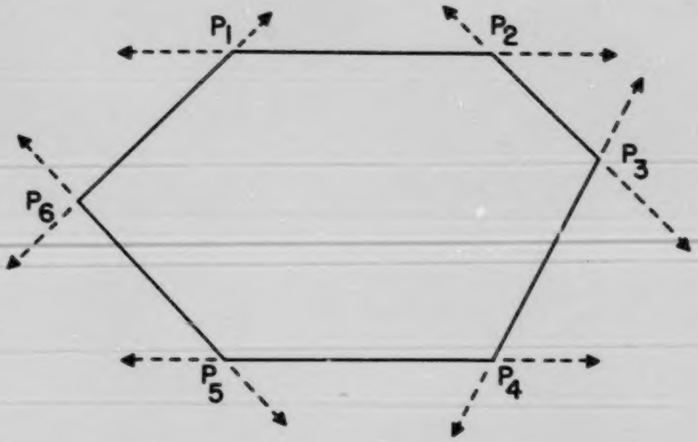
Then the union of the n segments is a polygon.

The n given points are vertices of the polygon, the n segments are sides of the polygon. By (2), any two segments with a common vertex determine an angle, which is called an angle of the polygon.

Notice that triangles are polygons of 3 vertices and 3 sides, and quadrilaterals are polygons of 4 vertices and 4 sides. Polygons of n vertices and n sides are sometimes called n-gons. Thus a triangle is a 3-gon and a quadrilateral is a 4-gon (although the terms 3-gon and 4-gon are almost never used.) 5-gons are called pentagons, 6-gons are hexagons. 8-gons are octagons, and 10-gons are decagons. The other n-gons, for reasonably small numbers n, also have special names taken from the Greek, but the rest of these special names are not very commonly used.

Each side of a polygon lies in a line, which separates the plane into two half-planes. If, for each side, the rest of the polygon lies entirely in one of the half-planes having that side on its edge, then the polygon is called a convex polygon.

Below is a convex polygon, with the lines drawn in to indicate why it is convex:

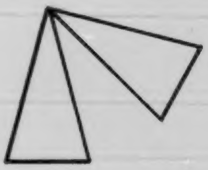


This is a natural term to use, because if a polygon is convex, it turns out that the polygon plus its interior forms a convex set in the sense that we defined long ago in Chapter 3. Just before the definition of a polygon, there are five examples of polygons. You should check that the first, second and fourth of these examples are convex polygons, but the third and fifth are not. You should check also that in the first, second and fourth cases, the polygon plus its interior forms a convex set, but that in the third and fifth cases this is not so.

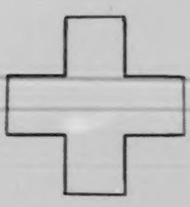
In this chapter we shall use polygons in the study of circles, to learn to calculate circumferences and areas. In the next chapter we shall calculate the volumes of prisms, pyramids, cones, and spheres. The basic procedure consists in approximating lengths and areas of curved figures with lengths and areas of polygonal figures, and seeing what happens as the approximations become better and better. A complete treatment of this last stage of the process is well beyond the subject matter of this course, but we will explain the logic of the situation as clearly as we can, and as completely as seems practical.

Problem Set 15-1

1. In the figure at the right, no three end-points are collinear and no two segments intersect except at their end-points. Nevertheless the figure is not a polygon. Why not?



2. Is the figure at the right a polygon? How many sides has it? How many vertices? What can you say about the relative lengths of the sides? About the measures of the angles?



*3. a. State a definition of the interior of a convex polygon. (Hint: Consider the definition of the interior of a triangle.)

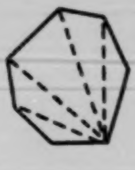
b. Make a sketch to illustrate that the union of a convex polygon and its interior is a polygonal region. (See definition of polygonal region in Chapter 11.)

4. A segment connecting two vertices of a polygon which are not end-points of the same side is a diagonal of the polygon.

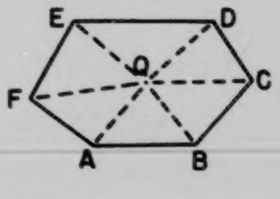
a. How many diagonals has a polygon with 3 sides? 4 sides? 5 sides? 6 sides? 103 sides? n sides?

b. Sketch a pentagon for which only two of the diagonals pass through its interior.

5. Use the figure at the right to show that the sum of the measures of the angles of a convex polygon of n sides is $S = (n - 2)180$.

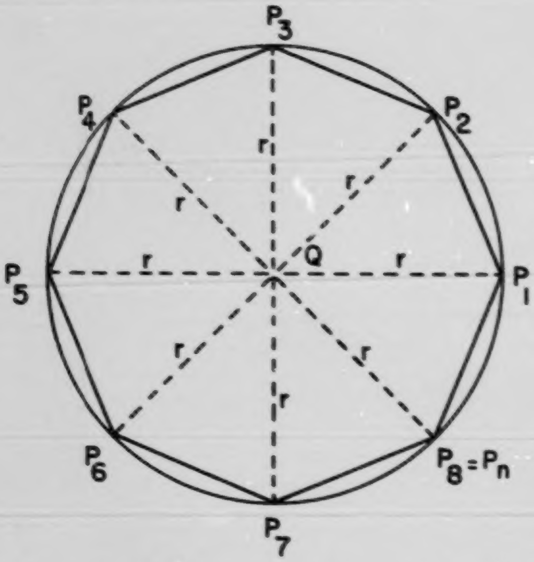


6. Verify the statement in the preceding problem, using this figure.



15-2. Regular Polygons.

Suppose we start with a circle, with center Q and radius r, and divide the circle into n congruent arcs, end to end. The figure shows the case $n = 8$.



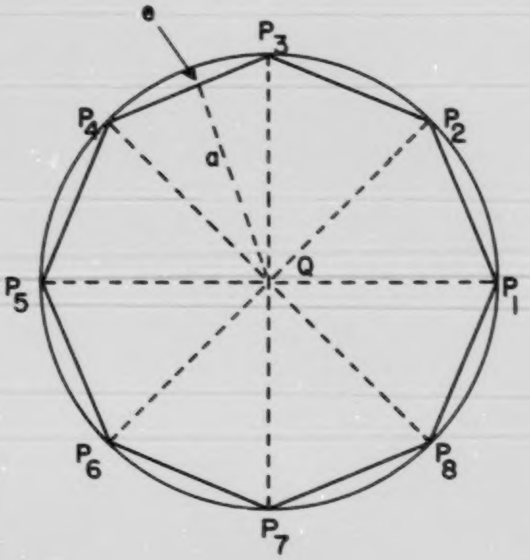
For each little arc, we draw the corresponding chord. This gives a polygon with vertices P_1, P_2, ..., P_n. The arcs are congruent, and so the chords (which are the sides of the polygon) are also congruent. If we draw segments from Q to each vertex of the polygon, we get a set of n isosceles triangles. In each triangle, $m\angle Q = \frac{360}{n}$, because $\frac{360}{n}$ is the measure of the intercepted arc in each case. Therefore all of the isosceles triangles are congruent. It follows that all of the angles of the polygon are congruent; the measure of an angle of the polygon is twice the measure of any base angle of any one of the isosceles triangles.

Thus the polygon has all of its sides congruent and all of its angles congruent.

Definitions: A convex polygon is regular if all its sides are congruent and all its angles are congruent. A polygon is inscribed in a circle if all of its vertices lie on the circle.

It is a fact that every regular polygon can be inscribed in a circle, but we will not stop to prove this, because we will not need it. We will be using regular polygons only in the study of circles, and all of the regular polygons that we will be talking about will be inscribed in circles by the method we have just described.

If P_1, P_2, ..., P_n is a regular polygon inscribed in a circle, then the triangles $\Delta\ P_1QP_2$, $\Delta\ P_2QP_3$, ..., are all congruent and they have the same base e and the same altitude a. These are shown, in the figure below, for $\Delta\ P_3QP_4$.



The area of each triangle is $\frac{1}{2}ae$, and therefore the total area of the regular n-gon is

$$A_n = n \cdot \frac{1}{2}ae = \frac{1}{2}ane.$$

Definition: The number a is called the apothem of the polygon. The sum of the lengths of the sides is called the perimeter.

We denote the perimeter by p. Thus, for a regular polygon, we have

$$p = n \cdot e.$$

In this notation, the area formula becomes

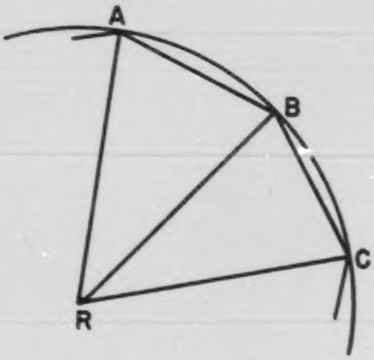
$$A_n = \frac{1}{2} a \cdot p.$$

Problem Set 15-2

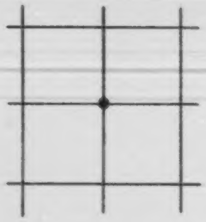
1. What is the ratio of the apothem of a square to its perimeter?
2. a. What size angle would be determined by drawing radii to the end-points of a side of a regular inscribed octagon?
   b. Use protractor and ruler to construct a regular octagon.
   c. Use compass and straight-edge to construct a regular octagon.
3. Use protractor and ruler to construct a regular pentagon.
4. A formula for the sum of the measures of the angles of any convex polygon of n sides is $(n - 2)180$. (See Problem 5 of Problem Set 15-1.) What would be a formula for the measure of each angle of a regular n-gon?
5. Is the polygon of Problem 2 in Problem Set 15-1 a regular 12-gon? Justify your answer.

6. The figure represents part of a regular polygon of which $\overline{AB}$ and $\overline{BC}$ are sides, and R is the center of the circle in which the polygon is inscribed. Copy and complete the table:

number of sides	$m\angle ARB$ or $m\angle BRC$	$m\angle ABR$ or $m\angle CBR$	$m\angle ABC$
3	___	___	___
4	___	___	___
5	___	___	___
6	___	___	___
___	45	___	___
9	40	70	140
___	___	___	144
12	___	___	___
15	___	___	___
18	___	___	___
20	___	___	___
24	___	___	___



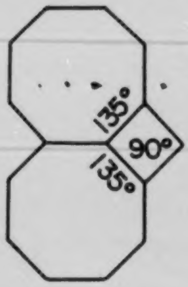
7. A plane can be covered by congruent square regions placed four at a vertex as shown.



a. How many equilateral triangles must be placed at a vertex to cover a plane?

b. What other class of regular polygonal regions can be used to cover a plane? How many would be needed at a vertex?

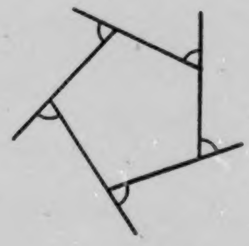
c. Two regular octagons and one square will completely cover the part of a plane around a point without any overlappings, as shown. What other combinations of three regular polygons (two of which are alike) will do this?
(Hint: Consider possible angle measures such as those listed in the last column of your table for Problem 6. Find solutions of the equation $2x + y = 360$ where x and y are angle measures for regular polygons having different numbers of sides. In the illustration $x = 135$ and $y = 90$.)



$$2 \cdot 135 + 90 = 360$$

d. Investigate the possibility of other coverings of a plane around a point by regular polygons.

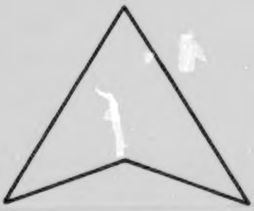
8. Show that the sum of the measures of the exterior angles of any convex polygon is 360.
(Hint: Count the supplements of the interior angles.)



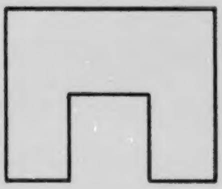
*9. a. A convex polygon of n sides (n is a positive even integer greater than 3) can be separated into how many quadrilateral regions by drawing diagonals from a given vertex?

b. Derive a formula for the sum of the measures of the angles of a convex polygon from your answer to part (a).

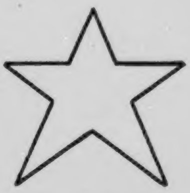
10. Let S be the sum of the measures of the angles of a polygon with n sides. If the polygon is convex, then $S = (n - 2)180$. In the following three figures, which are not convex, show that the formula is still correct if we regard S as the sum of the measures of the angles of the triangles into which each can be divided, assuming that no new vertices are introduced.



(a)

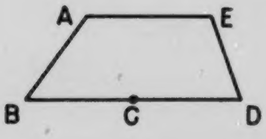


(b)



(c)

11. Show that in any polygon if an "artificial vertex" is inserted on one of the sides as shown so that the number of "sides" is increased by one, the formula for the angle sum still holds.



12. The sides of a regular hexagon are each 2 units long. If it is inscribed in a circle, find the radius of the circle and the apothem of the hexagon.

*13. A regular octagon with sides 1 unit long is inscribed in a circle. Find the radius of the circle.

15-3. The Circumference of a Circle. The Number π.

In this section and the next one we shall consider regular n-gons for various values of n. As usual, we denote the side, apothem, perimeter, etc. of a regular n-gon inscribed in a circle of radius r by e, a, p, etc.

Let C be the circumference of the circle that we have been discussing. It seems reasonable to suppose that if you want to measure C approximately, you can do it by inscribing a regular polygon with a large number of sides and then measuring the perimeter of the polygon. That is, the perimeter p ought to be a good approximation of C when n is large. Putting it another way, if we decide how close we want p to be to C, we ought to be able to get p to be this close to C merely by making n large enough. We describe this situation in symbols by writing

$$p \longrightarrow C,$$

and we say that p approaches C as a limit.

We cannot prove this, however; and the reason why we cannot prove it is rather unexpected. The reason is that so far, we have no mathematical definition of what is meant by the circumference of a circle. (We can't get the circumference merely by adding the lengths of certain segments, the way we did to get the perimeter of a polygon, because a circle doesn't contain any segments. Every arc of a circle, no matter how short you take the arc, is curved at least slightly.) But the remedy is easy: we take the statement

$$p \longrightarrow C$$

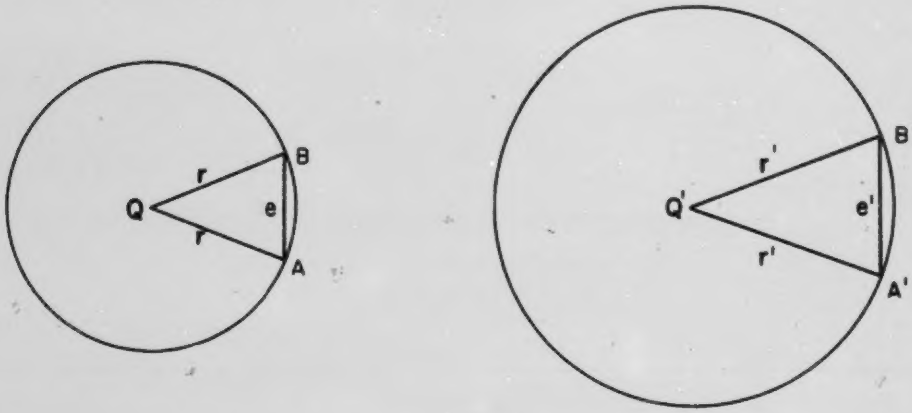
as our definition of C, thus:

Definition: The circumference of a circle is the limit of the perimeters of the inscribed regular polygons.

We would now like to go on, in the usual way, to define the number π as the ratio of the circumference of a circle to its diameter. But to make sure that this definition makes good sense, we first need to know that the ratio $\frac{C}{2r}$ is the same for all circles, regardless of their size. Thus we need to prove the following.

Theorem 15-1. The ratio $\frac{C}{2r}$, of the circumference to the diameter, is the same for all circles.

The proof is by similar triangles. Given a circle with center Q and radius r, and another circle, with center Q' and radius r', we inscribe a regular n-gon in each of them. (The same value of n must be used in each circle.)



In the figure we show only one side of each n-gon, with the associated isosceles triangle. Now $\angle AQB \cong \angle A'Q'B'$, because each of these angles has measure $\frac{360}{n}$. Therefore, since the adjacent sides are proportional,

$$\Delta AQB \sim \Delta A'Q'B'$$

by the S.A.S. Similarity Theorem. Therefore,

$$\frac{e}{r} = \frac{e'}{r'},$$

and so

$$\frac{p}{r} = \frac{p'}{r'}$$

where p is the perimeter ne of the first n-gon, and p' is the perimeter ne' of the second n-gon. Let C and C' be the circumferences of the two circles. Then $p \longrightarrow C$, by definition, and $p' \longrightarrow C'$, by definition. Therefore

$$\frac{C}{r} = \frac{C'}{r'}$$

and

$$\frac{C}{2r} = \frac{C'}{2r'},$$

which was to be proved.

The number $\frac{C}{2r}$, which is the same for all circles, is designated by π. We can therefore express the conclusion of Theorem 15-1 in the well-known form,

$$C = 2\pi r.$$

π is an irrational number and cannot be represented exactly in fractional form. It can however, be approximated as closely as we please by rational numbers. Some rational approximations to π are

$$3,\ 3.14,\ \frac{22}{7},\ 3.1416,\ \frac{355}{113},\ 3.14159265358979.$$

Problem Set 15-3

1. A regular polygon is inscribed in a circle, then another with one more side than the first is inscribed, and so on endlessly, each time increasing the number of sides by one.
   a. What is the limit of the length of the apothem?
   b. What is the limit of the length of a side?
   c. What is the limit of the measure of an angle?
   d. What is the limit of the perimeter of the polygon?
2. A certain tall person takes steps a yard long. He walks around a circular pond close to the edge taking 628 steps. What is the approximate radius of the pond? (Use 3.14 for π.)

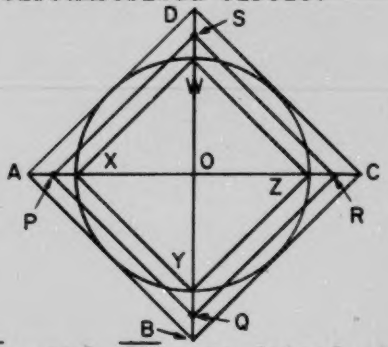
3. Which is the closer approximation to π, 3.14 or $\frac{22}{7}$?

4. The moon is about 240,000 miles from the earth, and its path around the earth is nearly circular. Find the circumference of the circle which the moon describes every month.

5. The earth is about 93,000,000 miles from the sun. The path of the earth around the sun is nearly circular. Find how far we travel every year "in orbit". What is our speed in this orbit in miles per hour.

6. The side of a square is 12 inches. What is the circumference of its inscribed circle? Of its circumscribed circle?

*7. In the figure, square XYZW is inscribed in circle O, and square ABCD is circumscribed about the circle. The diagonals of both squares lie in $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$. Given that a square PQRS is formed when the mid-points P, Q, R and S of $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$, and $\overline{DW}$ are joined, is the perimeter of this square equal to, greater than, or less than the circumference of circle O? Let OX = 1 and justify your answer by computation.



8. The radius of a circle is 10 feet. By how much is its circumference changed if its radius is increased by 1 foot? If the radius were originally 1000 feet, what would be the change in the circumference when the radius is increased by 1 foot?

15-4. Area of a Circle.

In Chapter 11 we considered areas of polygonal regions, defined in terms of a basic region, the triangular region, which is the union of a triangle and its interior. In talking about areas associated with a circle we make a similar basic definition.

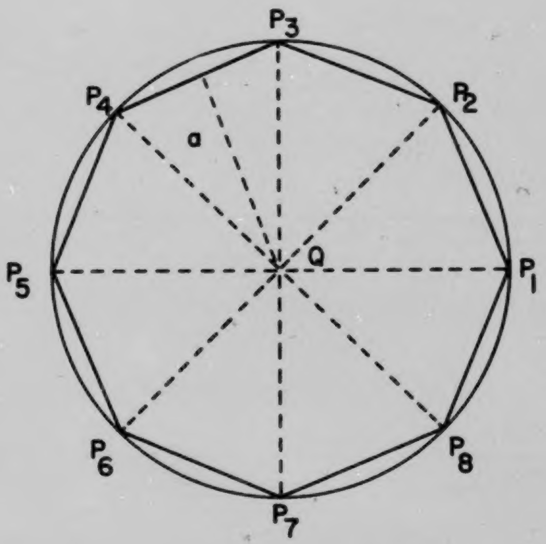
Definition: A circular region is the union of a circle and its interior.

In speaking of "the area of a triangular region" we found it convenient to abbreviate this phrase to "the area of a triangle". Similarly, we shall usually say "the area of a circle" as an abbreviation of "the area of a circular region".

We shall now get a formula for the area of a circle. We already have a formula for the area of an inscribed regular n-gon; this is

$$A_n = \frac{1}{2} ap$$

where a is the apothem and p is the perimeter.



In this situation there are three quantities involved, each depending on n. These are p, a and A_n. To get our formula for the area of a circle, we need to find out what limits these quantities approach as n becomes very large.

(1) What happens to A_n. A_n is always slightly less than the area A of the circle, because there are always some points that lie inside the circle but outside the regular n-gon. But the difference between A_n and A is very small when n is very large, because when n is very large the polygon almost fills up the interior of the circle. Thus, we expect that

$$A_n \longrightarrow A.$$

But just as in the case of the circumference of the circle, this can never be proved, since we have not yet given any definition of the area of a circle. Here also, the way out is easy:

Definition: The area of a circle is the limit of the areas of the inscribed regular polygons.

Thus, $A_n \longrightarrow A$ by definition.

(2) What happens to a. The apothem a is always slightly less than r, because either leg of a right triangle is shorter than the hypotenuse. But the difference between a and r is very small when n is very large. Thus,

$$a \longrightarrow r.$$

(3) What happens to p. By definition of C, we have $p \longrightarrow C$.

Fitting together the results in (2) and (3), we get

$$\tfrac{1}{2}ap \longrightarrow \tfrac{1}{2}rC.$$

Therefore $$A_n \longrightarrow \tfrac{1}{2}rC.$$

But we knew from (1) that $A_n \longrightarrow A$. Therefore

$$A = \tfrac{1}{2}rC.$$

Combining this with the formula $C = 2\pi r$ gives

$$A = \pi r^2.$$

Thus the formula that you have known for years finally becomes a theorem:

Theorem 15-2. The area of a circle of radius r is πr^2.

Problem Set 15-4

1. Find the circumference and area of a circle with radius

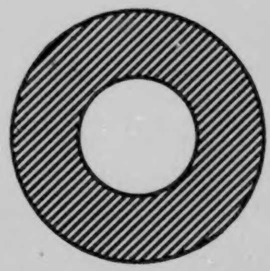
   a. 5. b. 10.

2. Find the circumference and area of a circle with radius

   a. n. b. $10n$.

3. a. Find the area of one face of this iron washer if its diameter is 4 centimeters and the diameter of the hole is 2 centimeters.

   b. Would the area be changed if the two circles were not concentric?



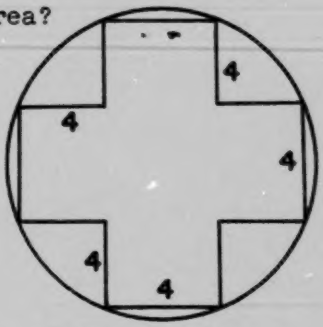
4. The radius of the larger of two circles is three times the radius of the smaller. Compare the area of the first to that of the second.

5. The circumference of a circle and the perimeter of a square are each equal to 20 inches. Which has the greater area? How much greater is it?

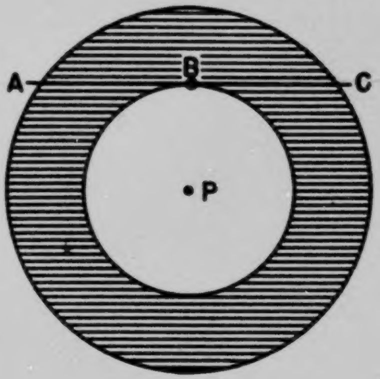
6. Given a square whose side is 10 inches, what is the area between its circumscribed and inscribed circles?

7. An equilateral triangle is inscribed in a circle. If the side of the triangle is 12 inches, what is the radius of the circle? The circumference? The area?

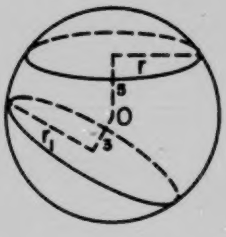
8. The cross inside the circle is divisible into 5 squares. Find the area which is inside the circle and outside the cross.



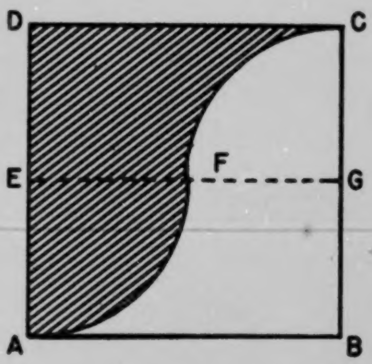
9. Given: Two concentric circles with center P, $\overline{AC}$ is a chord of the larger and is tangent to the smaller at B. Prove: The area of the ring (annulus) is πBC^2.



10. In a sphere whose radius is 10 inches, sections are made by planes 3 inches and 5 inches from the center. Which section will be the larger? Prove that your answer is correct.

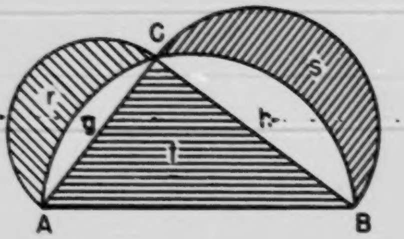


*11. In the figure, ABCD is a square in which E, F, G are mid-points of $\overline{AD}$, $\overline{AC}$, and $\overline{CB}$, respectively. $\widehat{AF}$ and $\widehat{FC}$ are circular arcs with centers E and G respectively. If the side of the square is s, find the area of the shaded portion.

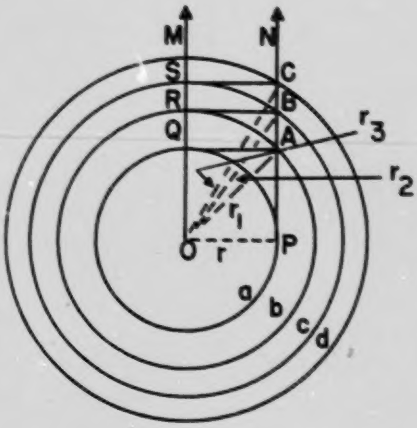


*12. In the figure, semi-circles are drawn with each side of right triangle ABC as diameter. Areas of each region in the figure are indicated by lower case letters.

Prove: $r + s = t$.



*13. A special archery target, with which an amateur can be expected to hit the bulls-eye as often as any ring, is constructed in the following way. Rays $\overrightarrow{OM}$ and $\overrightarrow{PN}$ are parallel. A circle with center O and radius r equal to the distance between the rays, is drawn intersecting $\overrightarrow{OM}$ at Q. $\overline{QA} \perp \overrightarrow{QM}$. Then a circle with center O and radius OA, or r_1 is drawn. This process is repeated by drawing perpendiculars at R and at S, and circles with radii OB and OC. Note that we arbitrarily stop at four concentric circles.



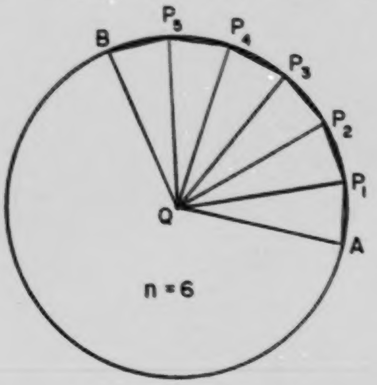
a. Find r_1, r_2, r_3 in terms of r.

b. Show that the areas of the inner circle and the three "rings", represented by a, b, c, and d, are equal.

14. An isosceles trapezoid whose bases are 2 inches and 6 inches is circumscribed about a circle. Find the area of the portion of the trapezoid which lies outside the circle.

15-5. Lengths of Arcs. Areas of Sectors.

Just as we define the circumference of a circle as the limit of the perimeters of inscribed regular polygons, so we can define the length of a circular arc as a suitable limit.



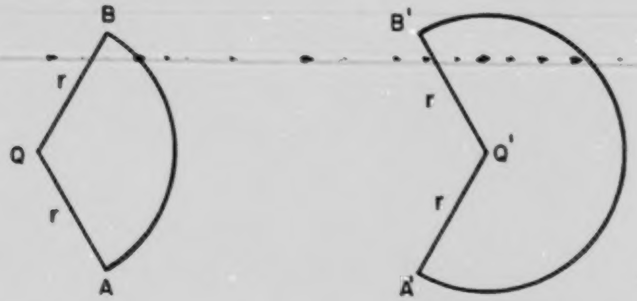
If $\widehat{AB}$ is an arc of a circle with center Q, we take points P_1, P_2, ... P_{n-1} on $\widehat{AB}$ so that each of the n angles $\angle AQP_1$, $\angle P_1QP_2$, ..., $\angle P_{n-1}QB$ has measure $\frac{1}{n} \cdot m\widehat{AB}$.

Definition: The length of arc $\widehat{AB}$ is the limit of $AP_1 + P_1P_2 + \dots + P_{n-1}B$ as we take n larger and larger.

It is convenient, in discussing lengths of arcs, to consider an entire circle as an arc of measure 360. Any point of the circle can be considered as the coincident end-points of the arc. The circumference of a circle can then be considered to be simply the length of an arc of measure 360.

The basic theorem on arc length is the following:

Theorem 15-3. If two arcs have equal radii, then their lengths are proportional to their measures.



$$\frac{\text{length } \widehat{AB}}{m\widehat{AB}} = \frac{\text{length } \widehat{A'B'}}{m\widehat{A'B'}}.$$

The proof of this theorem is very hard, and quite unsuitable for a beginning geometry course. We make no attempt to prove it here, but, like Theorem 13-6 (to which it is closely related), treat it as if it were a new postulate.

Theorems 15-1 and 15-3 can be combined to give a general formula for the length of an arc.

Theorem 15-4. An arc of measure q and radius r has length $\frac{\pi}{180}qr$.

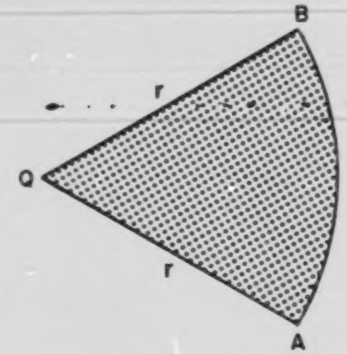
Proof: If C is the circumference of a circle of radius r we have, by Theorem 15-3,

$$\frac{L}{q} = \frac{C}{360}.$$

By Theorem 15-1, $C = 2\pi r$. Substituting this value of C above and solving for L gives

$$L = \frac{\pi}{180}qr.$$

A sector of a circle is a region bounded by two radii and an arc, like this:



More precisely:

Definitions: If $\widehat{AB}$ is an arc of a circle with center Q and radius r, then the union of all segments $\overline{QP}$, where P is any point of $\widehat{AB}$, is a sector. $\widehat{AB}$ is the arc of the sector and r is the radius of the sector.

The following theorem is proved just like Theorem 15-2.

Theorem 15-5. The area of a sector is half the product of its radius by the length of its arc.

Combined with Theorem 15-4, we get

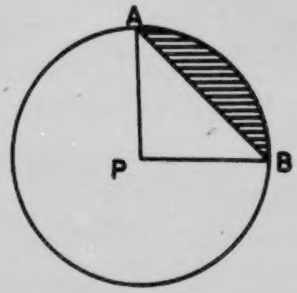
Theorem 15-6. The area of a sector of radius r and arc measure q is $\frac{\pi}{360} qr^2$.

Problem Set 15-5

1. The radius of a circle is 15 inches. What is the length of an arc of 60°? of 90°? of 72°? of 36°?
2. The radius of a circle is 6. What is the area of a sector with an arc of 90°? of 1°?
3. If the length of a 60° arc is one centimeter, find the radius of the arc. Also find the length of the chord of the arc.

4. In a circle of radius 2, a sector has area π. What is the measure of its arc?

5. A segment of a circle is the region bounded by a chord and an arc of the circle. The area of a segment is found by subtracting the area of the triangle formed by the chord and the radii to its end-points from the area of the sector.



In the figure, $m\angle APB = 90$. If $PB = 6$, then

Area of sector $PAB = \frac{1}{4}\pi \cdot 6^2 = 9\pi$.

Area of triangle $PAB = \frac{1}{2} \cdot 6^2 = 18$.

Area of segment $= 9\pi - 18$ or approx. 10.26.

Find the area of the segment if:

a. $m\angle APB = 60$; $r = 12$.

b. $m\angle APB = 120$; $r = 6$.

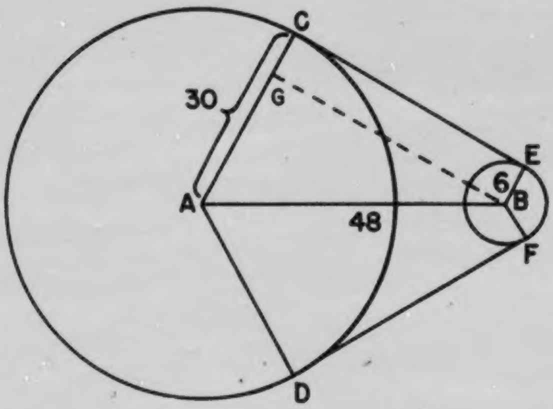
c. $m\angle APB = 45$; $r = 8$.

6. If a wheel of radius 10 inches rotates through an angle of 36°,

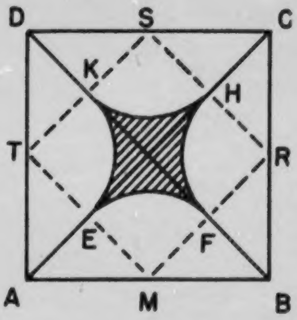
a. how many inches does a point on the rim of the wheel move?

b. how many inches does a point on the wheel 5 inches from the center move?

7. A continuous belt runs around two wheels of radius 6 inches and 30 inches. The centers of the wheels are 48 inches apart. Find the length of the belt.

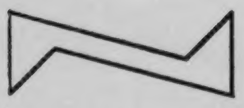


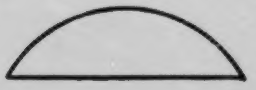
8. In this figure ABCD is a square whose side is 8 inches. With the mid-points of the sides of the square as centers, arcs are drawn tangent to the diagonals. Find the area enclosed by the four arcs.

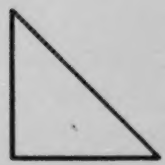


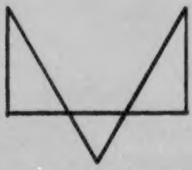
Review Problems

1. Which of the figures below are polygons? Which ones are convex polygons?









2. Does every regular polygon have
   a. each side congruent to every other side?
   b. each angle congruent to every other angle?
   c. at least two sides parallel?

3. What is the measure of an angle of a regular
   a. pentagon?
   c. octagon?
   b. hexagon?
   d. decagon?

4. If the measure of an angle of a regular polygon is 150, how many vertices does the polygon have?

5. a. If both a square and a regular octagon are inscribed in the same circle, which has the greater apothem? the greater perimeter?
   b. Answer the same questions for circumscribed figures.

6. From what formula relating to regular polygons is the formula for the area of a circle derived?

7. If C is the circumference of a circle and r is its radius, what is the value of $\frac{C}{r}$?

8. If the circumference of a circle is 12 inches, the length of its radius will lie between what two consecutive integers?

9. Find the measure of an exterior angle of

   a. a regular pentagon. b. a regular n-gon.

10. What is the radius of a circle if its circumference is equal to its area?

11. If the radius of one circle is 10 times the radius of another, give the ratio of

   a. their diameters. c. their areas.

   b. their circumferences.

12. If a regular hexagon is inscribed in a circle of radius 5, what is the length of each side? What is the length of the arc of each side?

13. Show that the area of a circle is given by the formula $A = \frac{1}{4}\pi d^2$, where d is the diameter of the circle.

14. A wheel has a 20 inch diameter. How far will it roll if it turns 270°?

15. The angle of a sector is 10° and its radius is 12 inches. Find the area of the sector and the length of its arc.

16. Prove that the area of an equilateral triangle circumscribed about a circle is four times the area of an equilateral triangle inscribed in the circle.

*17. This problem came up in a college zoology course: Two woodchucks dig burrows at a distance r from each other, and each of them is the nearest neighbor of the other. If a third woodchuck moves into the region, how large is the area in which he can settle so that he will become the nearest neighbor of each of the original woodchucks?

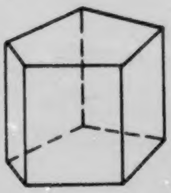
18. One regular 7-sided polygon has area 8 and another regular 7-sided polygon has area 18. What is the ratio of a side of the smaller to a side of the larger?

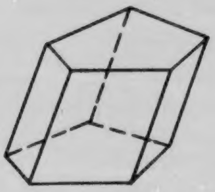
Chapter 16

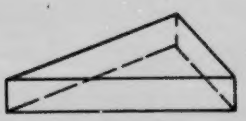
VOLUMES OF SOLIDS

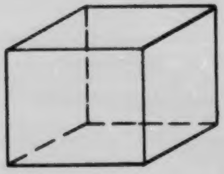
16-1. Prisms.

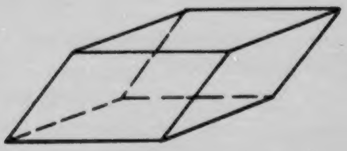
Here are some pictures of prisms:







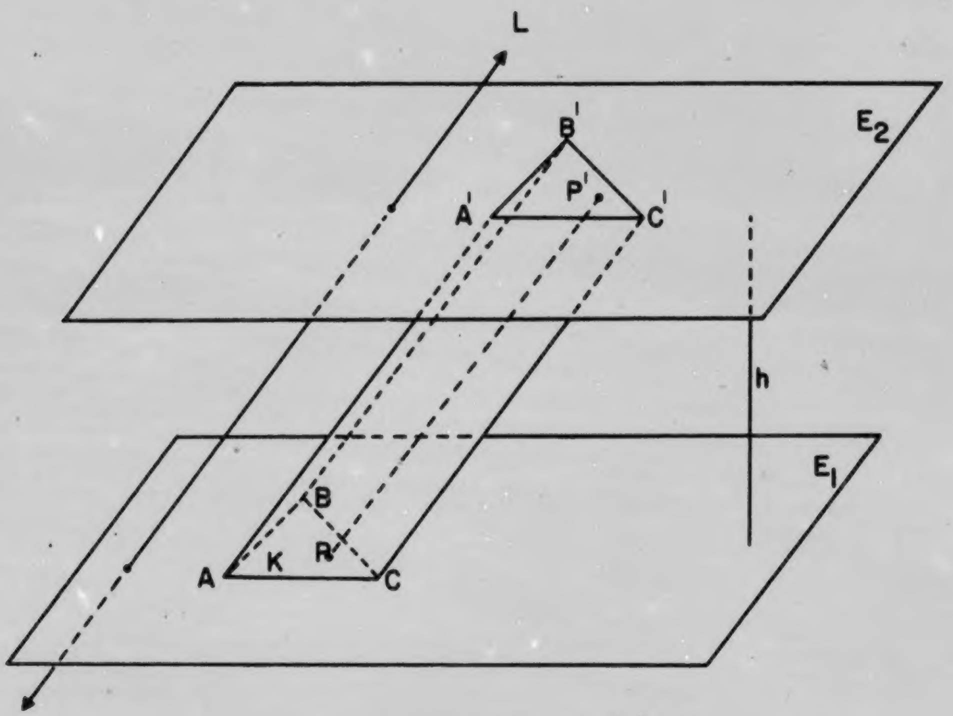




A prism can be thought of as the solid swept out in moving a polygonal region parallel to itself from one position to another. In this process each point of the region describes a line segment, and these segments are all parallel to one another. The prism itself can be thought of as just the set of all such line segments, as if it were made up of a bundle of parallel wires.

These considerations lead us to the following precise definition.

Definitions. Let E_1 and E_2 be two parallel planes, L a transversal, and K a polygonal region in E_1 which does not intersect L. For each point P of K let $\overline{PP'}$ be a segment parallel to L with P' in E_2. The union of all such segments is called a prism.

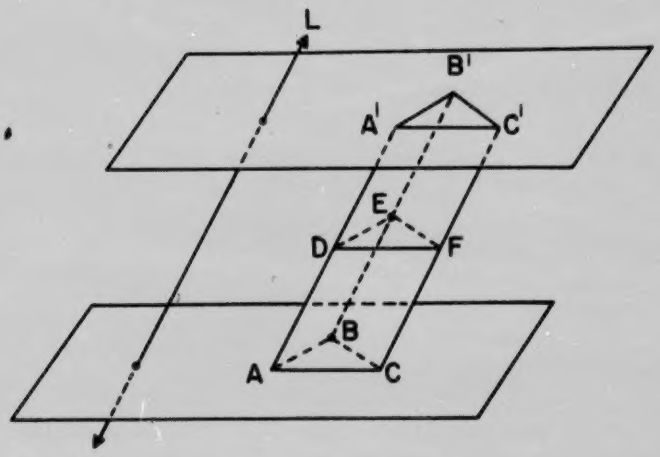


Definitions. The polygonal region K is called the lower base, or just the base, of the prism. The set of all the points P', that is, the part of the prism that lies in E_2, is called the upper base. The distance h between E_1 and E_2 is the altitude of the prism. If L is perpendicular to E_1 and E_2 the prism is called a right prism.

Prisms are classified according to their bases: a triangular prism is one whose base is a triangular region, a rectangular prism is one whose base is a rectangular region, and so on.

Definition. A cross-section of a prism is its intersection with a plane parallel to its base, provided this intersection is not empty.

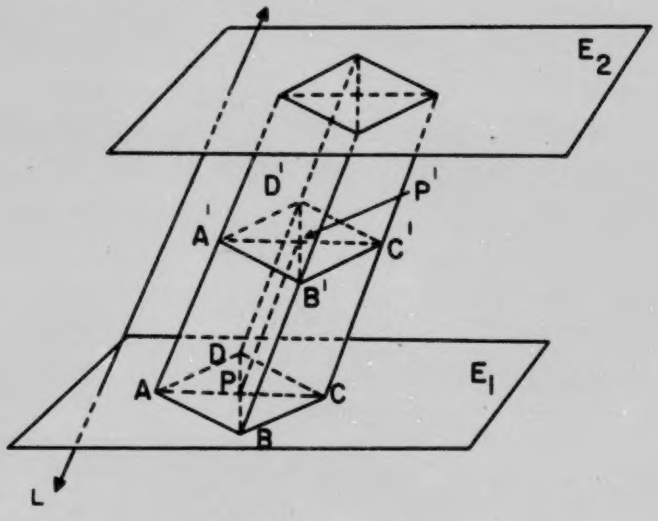
Theorem 16-1. All cross-sections of a triangular prism are congruent to the base.



Proof: Let the triangular region ABC be the base of a prism, and let a cross-section plane intersect $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ in D, E and F. $\overline{AD} \parallel \overline{BE}$ by definition of a prism, and $\overline{AB} \parallel \overline{DE}$ by Theorem 10-1. Hence, ABED is a parallelogram, and so DE = AB, because opposite sides of a parallelogram are congruent. Similarly, DF = AC and EF = BC. By the S.S.S. Theorem, $\Delta DEF \cong \Delta ABC$.

Corollary 16-1-1. The upper and lower bases of a triangular prism are congruent.

Theorem 16-2. (Prism Cross-Section Theorem.) All cross-sections of a prism have the same area.



Proof: By definition of a polygonal region, the base can be cut up into triangular regions. Thus the prism is cut up into triangular prisms whose bases are the triangular regions.

By Theorem 16-1, each triangle in the base is congruent to the corresponding triangle in the cross-section. (Thus, in the figure, $\Delta\ PAB \cong \Delta\ P'A'B'$, $\Delta\ PBC \cong \Delta\ P'B'C'$, and so on.) The area of the base is the sum of the areas of the triangular regions in the base; and the area of the cross-section is the sum of the areas of the corresponding triangular regions in the cross-section. Since congruent triangles have the same area, the theorem follows.

Corollary 16-2-1. The two bases of a prism have equal areas.

(Note: Since we have not defined congruence for figures more complicated than triangles, Theorem 16-2, while intuitively clear, must be proved using our available definitions. However, it is evident that with any reasonable general definition of congruence between geometric figures the theorem should hold for any prism. In Appendix VIII such a definition of congruence is given, and then the proof of Theorem 16-1 needs only a slight modification to prove that a cross-section of any prism is congruent to the base.)

Ordinarily we are concerned only with convex prisms, that is, prisms whose bases are convex polygonal regions. We can therefore speak of a "side" or a "vertex" of the base.

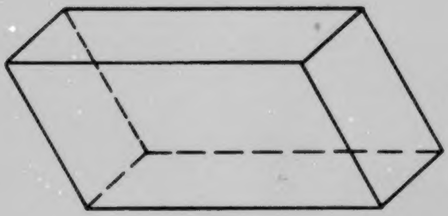
In the following definitions the notation is the same as that for the original definition of a prism.

Definitions: A lateral edge of a prism is a segment $\overline{AA'}$, where A is a vertex of the base of the prism. A lateral face is the union of all segments $\overline{PP'}$ for which P is a point in a given side of the base. The lateral surface of a prism is the union of its lateral faces. The total surface of a prism is the union of its lateral surface and its bases.

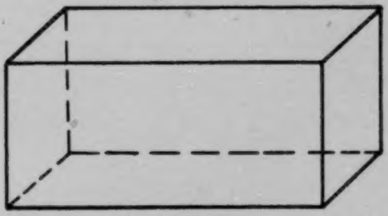
Theorem 16-3. The lateral faces of a prism are parallelogram regions, and the lateral faces of a right prism are rectangular regions.

A formal proof involves a discussion of separation properties and is rather long and tedious. While you may want to work out a formal proof, you can convince yourself of the correctness of the theorem by applying the definitions of prism and lateral face to the diagram for Theorem 16-1 or 16-2.

Definitions: A parallelepiped is a prism whose base is a parallelogram region. A rectangular parallelepiped is a right rectangular prism.



Parallelepiped

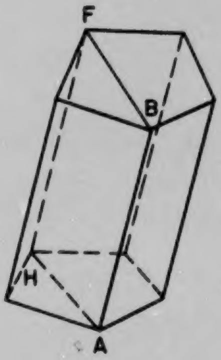


Rectangular Parallelepiped

Note: While in the preceding theorem and definitions we have been careful to refer to the base and the cross-section of a prism as regions, we will often use base and cross-section to mean the polygon which bounds the region and conversely, the context will make clear the intended use.

Problem Set 16-1

1. Prove that two non-adjacent lateral edges of a prism are coplanar, and that the intersection of their plane and the prism is a parallelogram. (Hint: For the figure shown, prove ABFH is a parallelogram.)



2. Find the area of the lateral surface of a right prism whose altitude is 10 if the sides of the pentagonal base are 3, 4, 5, 7, 2.

3. Find the area of the total surface of a right triangular prism if the base is an equilateral triangle 8 inches on a side and the height of the prism is 10 inches.

4. Prove that the lateral area (area of the lateral surface) of a right prism is the product of the perimeter of its base and the length of a lateral edge.

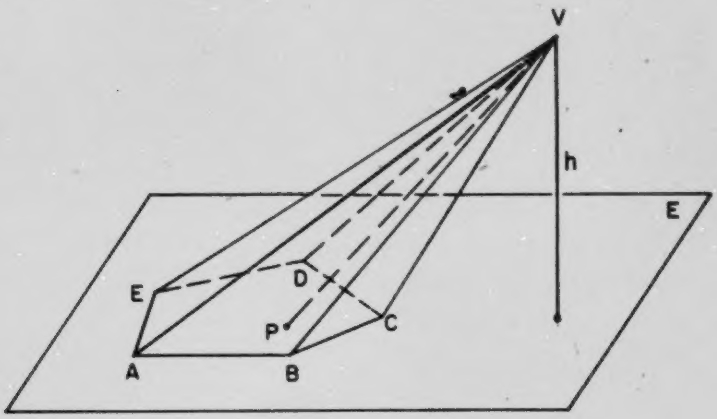
5. If the sides of a cross-section of a triangular prism are 3, 6, and $3\sqrt{3}$, then any other cross-section will be a triangle whose sides are _____, ______, and _______, whose angles measure ______, ______, ________, and whose area is ______.

6. The length of a lateral edge of a right prism is 10 inches and its lateral area is 52 square inches. What is the perimeter of its base?

16-2. Pyramids.

Pyramids are quite similar to prisms in some respects. In particular many of the terms carry over, and we shall use some of them without formal definition.

Definitions: Let K be a polygonal region in a plane E, and V a point not in E. For each point P in K there is a segment $\overline{PV}$. The union of all such segments is called a pyramid with base K and vertex V. The distance h from V to E is the altitude of the pyramid.

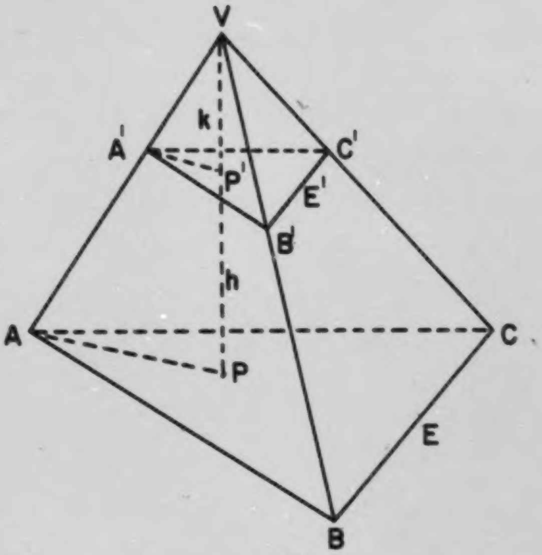


The next two theorems are analogous to Theorems 16-1 and 16-2.

Theorem 16-4. A cross-section of a triangular pyramid, by a plane between the vertex and the base, is a triangular region similar to the base. If the distance from the vertex to the cross-section plane is k and the altitude is h, then the ratio of the area of the cross-section to the area of the base is $(\frac{k}{h})^2$.

Restatement: Let Δ ABC be in plane E and point V a distance h from E. Let plane $\overline{E'}$, $\overline{\text{parallel}}$ to E .and at distance k from V, intersect $\overline{VA}$, $\overline{VB}$, $\overline{VC}$ in A', B', C'. Then $\Delta A'B'C' \sim \Delta ABC$, and .

$$\frac{\text{area } \Delta A'B'C'}{\text{area } \Delta ABC} = (\frac{k}{h})^2 .$$



Proof: Let $\overline{VP} \perp E$ and let $\overline{VP}$ intersect E' in P'. Then $h = VP$, $k = VP'$.

(1) $\overline{AP} \parallel \overline{A'P'}$ by Theorem 10-1.

$\triangle VA'P' \sim \triangle VAP$ by Corollary 12-3-2.

$\frac{VA'}{VA} = \frac{VP'}{VP} = \frac{k}{h}$ by definition of similar triangles.

(2) $\overline{A'B'} \parallel \overline{AB}$ by Theorem 10-1.

$\triangle VA'B' \sim \triangle VAB$ by Corollary 12-3-2.

$\frac{A'B'}{AB} = \frac{VA'}{VA} = \frac{k}{h}$ by (1) and definition.

(3) Similarly,

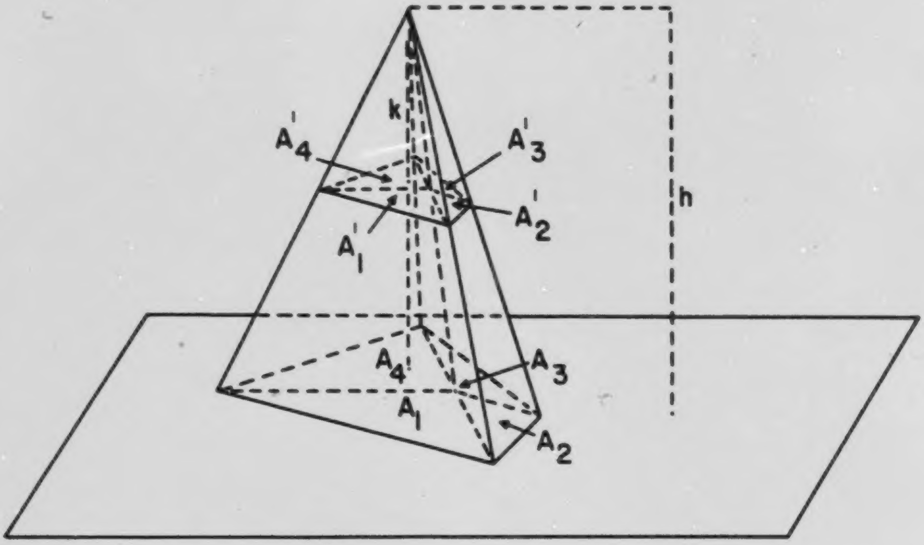
$\frac{B'C'}{BC} = \frac{k}{h}$, $\frac{C'A'}{CA} = \frac{k}{h}$.

(4) From (2) and (3)

$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = \frac{k}{h}$.

Therefore $\triangle A'B'C' \sim \triangle ABC$ by the S.S.S. Similarity Theorem, and $\frac{\text{area } \triangle A'B'C'}{\text{area } \triangle ABC} = \left(\frac{k}{h}\right)^2$ by Theorem 12-7.

Theorem 16-5. In any pyramid, the ratio of the area of a cross-section and the area of the base is $(\frac{k}{h})^2$, where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross-section.



Proof: Let us cut up the base into triangular regions with areas $A_1, A_2, \ldots, A_n$. (In the figure, $n = 4$.) Let $A_1', A_2', \ldots, A_n'$ be the areas of the corresponding triangular regions in the cross-section. Let A be the area of the base, and let A' be the area of the cross-section. Then

$$A = A_1 + A_2 + \ldots + A_n ,$$

and

$$A' = A_1' + A_2' + \ldots + A_n'.$$

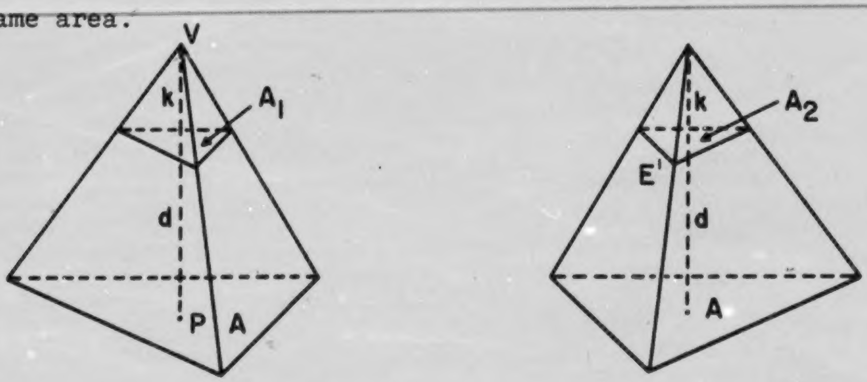
By the result which we have just proved for triangular pyramids, we know that $A_1' = (\frac{k}{h})^2 A_1$, $A_2' = (\frac{k}{h})^2 A_2$, and so on. Therefore

$$A' = (\frac{k}{h})^2 (A_1 + A_2 + \ldots + A_n)$$
$$= (\frac{k}{h})^2 A.$$

Therefore $\frac{A'}{A} = (\frac{k}{h})^2$, which was to be proved.

Theorem 16-5 has the following consequence.

Theorem 16-6. (The Pyramid Cross-Section Theorem.) Given two pyramids with the same altitude. If the bases have the same area, then cross-sections equidistant from the bases also have the same area.



In the figure, for the sake of simplicity, we show triangular pyramids, but the proof does not depend on the shape of the base.

Let A be the area of each of the bases, and let A_1 and A_2 be the areas of the cross-sections. Let h be the altitude of each of the pyramids, and let d be the distance between each cross-section and the corresponding base. Then the vertices of the two pyramids are at the same distance $k = h - d$ from the planes of the cross-sections. Therefore

$$\frac{A_1}{A} = (\frac{k}{h})^2 = \frac{A_2}{A},$$

by the previous theorem. Since the denominators on the left and right are equal, so also are the numerators. Therefore, $A_1 = A_2$, which was to be proved.

Problem Set 16-2

1. If the base of a pyramid is a square, each cross-section will be a ______. If the base of a pyramid is an equilateral triangle whose side is 9, each cross-section will be ______________ and the length of a side of the cross-section one-third of the distance from the vertex to the base will be _______.

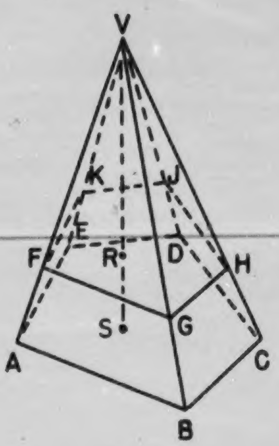
2. Given two pyramids, one triangular, one hexagonal, with equal base areas. In each the altitude is 6 inches. The area of a cross-section of the triangular pyramid, 2 inches from the base, is 25 square inches. What is the area of a cross-section 2 inches from the base of the hexagonal pyramid?

3. A regular pyramid is a pyramid whose base is a regular polygonal region having for its center the foot of the perpendicular from the vertex to the base.
Prove that the lateral faces of a regular pyramid are bounded by congruent isosceles triangles.

*4. Given a triangular pyramid with vertex V and base ABC, find a plane whose intersection with the pyramid is a parallelogram.

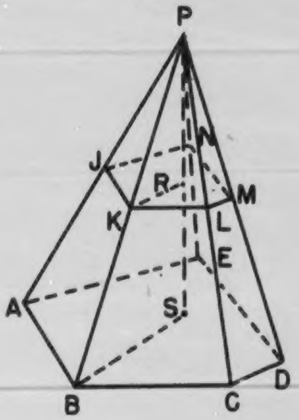
5. Show that the lateral area of a regular pyramid is given by $A = \frac{1}{2}ap$ in which p is the perimeter of the base and a is the altitude of a lateral face.

6. FGHJK is parallel to base ABCDE in the pyramid shown here, with altitude VS = 7 inches and altitude VR = 4 inches. If the area of ABCDE is 336 square inches, what is the area of FGHJK?



7. A regular pyramid has a square base, 10 inches on a side, and is one foot tall. Find the lateral area of the pyramid and the area of the cross-section 3 inches above the base.

*8. Prove: In any pyramid, the ratio of the area of a cross-section to the area of the base is $(\frac{a}{b})^2$, where a is the length of a lateral edge of the smaller pyramid and b is the corresponding lateral edge of the larger pyramid. (Hint: Draw altitude $\overline{PS}$.)

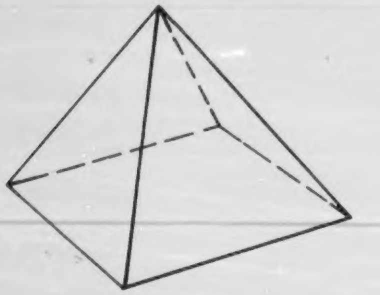


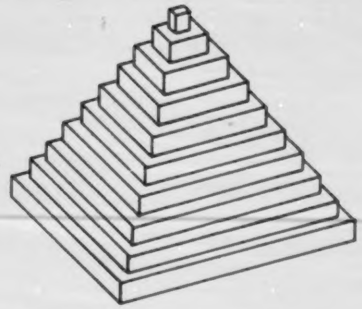
16-3. Volumes of Prisms and Pyramids, Cavalieri's Principle.

A vigorous treatment of volumes requires a careful definition of something analogous to polygonal regions in a plane (*polyhedral regions* is the name) and the introduction of postulates similar to the four area postulates. We will not give such a treatment, but instead will rely on your intuition to a considerable extent, particularly when it comes to cutting up solids or fitting them together. However, we will state explicitly the two *numerical* postulates we need. One of them is the analog of Postulate 20, which gave the area of a rectangle.

> *Postulate 21*. The volume of a rectangular parallelepiped is the product of the altitude and the area of the base.

To understand what is going on in our next postulate, let us first think of a physical model. We can make an approximate model of a square pyramid by forming a stack of thin cards, cut to the proper size, like this:

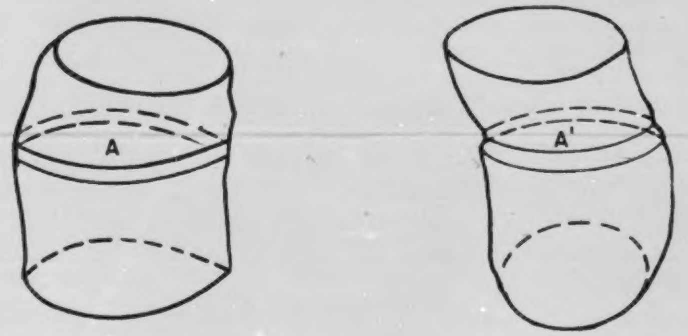




The figure on the left represents the exact pyramid, and the figure on the right is the approximate model made from cards.

Now suppose we drill a narrow hole in the model, from the top to some point of the base, and insert a thin rod so that it goes through every card in the model. We can then tilt the rod in any way we want, keeping its bottom end fixed on the base. The shape of the model then changes, but its volume does not change. The reason is that its volume is simply the total volume of the cards; and this total volume does not change as the cards slide along each other.

The same principle applies more generally. Suppose we have two solids with bases in a plane which we shall think of as horizontal. If all horizontal cross-sections of the two solids at the same level have the same area then the two solids have the same volume.



A = A'

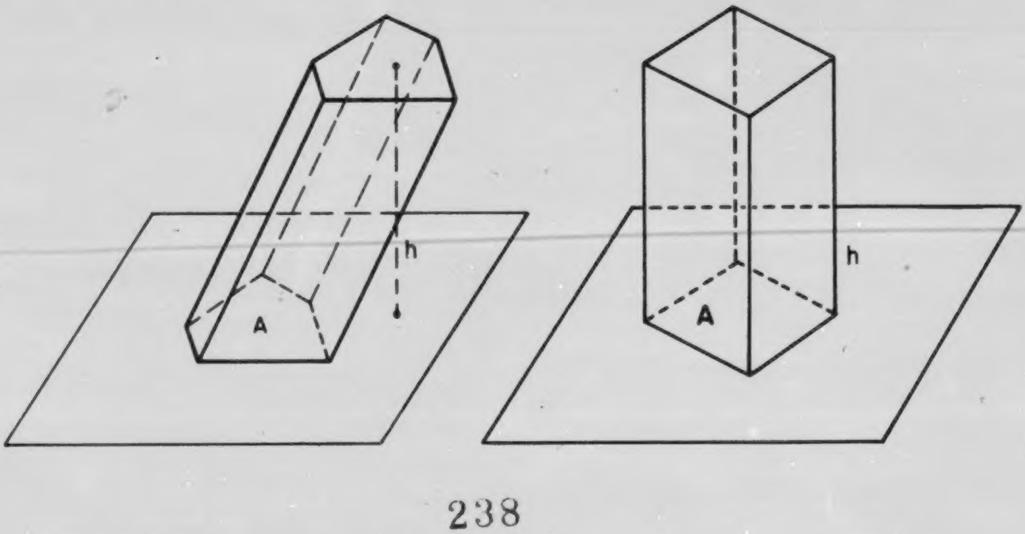
The reason is that if we make a card-model of each of the solids, then each card in the first model has exactly the same volume as the corresponding card in the second model. Therefore the volumes of the two models are exactly the same. The approximation given by the models is as close as we please, if only the cards are thin enough. Therefore the volumes of the two solids that we started with are the same.

The principle involved here is called *Cavalieri's Principle*. We haven't proved it; we have merely been explaining why it is reasonable. Let us therefore state it in the form of a postulate:

Postulate 22. (Cavalieri's Principle.) Given two solids and a plane. If for every plane which intersects the solids and is parallel to the given plane the two intersections have equal areas, then the two solids have the same volume.

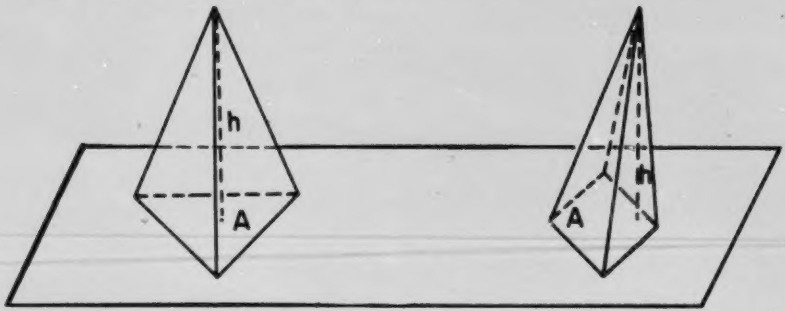
Cavalieri's Principle is the key to the calculation of volumes, as we shall soon see.

Theorem 16-7. The volume of any prism is the product of the altitude and the area of the base.



Proof: Let h and A be the altitude and the base area of the given prism. Consider a rectangular parallelepiped with the same altitude h and the base area A, and with its base in the same plane as the base of the given prism. We know by the Prism Cross-Section Theorem that all cross-sections, for both prisms, have the same area A. By Cavalieri's Principle, this means that they have the same volume. Since the volume of the rectangular parallelepiped is Ah by Postulate 21, the theorem follows.

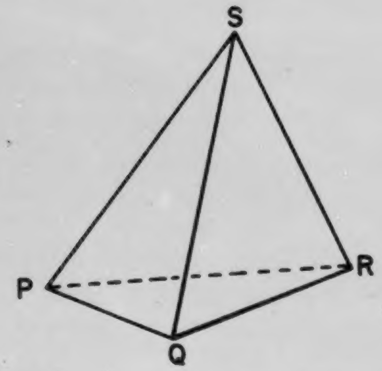
Theorem 16-8. If two pyramids have the same altitude and the same base area, then they have the same volume.

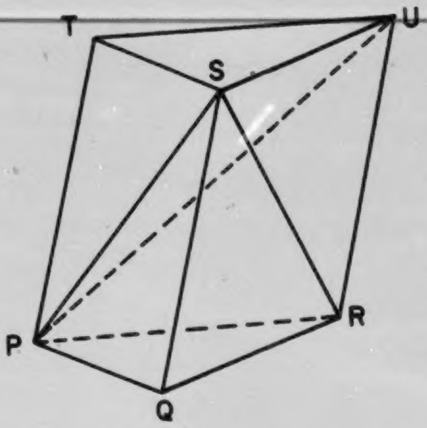


Proof: By the Pyramid Cross-Section Theorem, corresponding cross-sections of the two pyramids have the same area. By Cavalieri's Principle, this means that the volumes are the same.

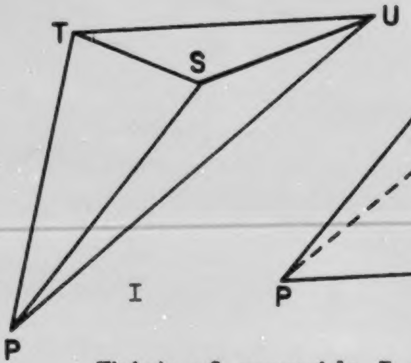
Theorem 16-9. The volume of a triangular pyramid is one-third the product of its altitude and its base area.

Proof: Given a triangular pyramid with base PQR and vertex S, we take a triangular prism PQRTSU with the same base and altitude, like this:





We next cut the prism into three triangular pyramids, one of them being the original one, like this:



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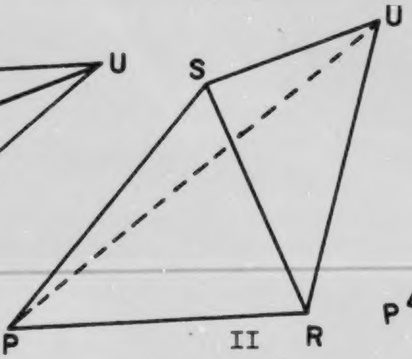
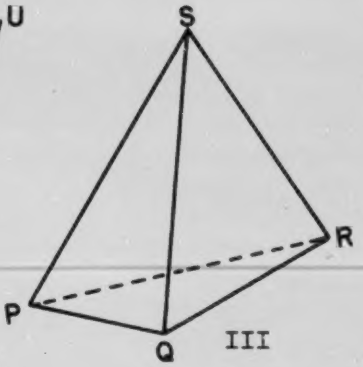


II



III

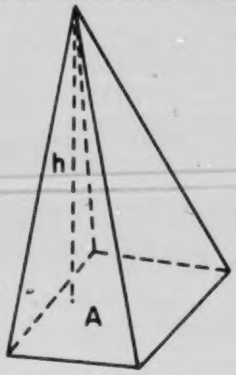
Think of pyramids I and II as having bases PTU and PRU, and common vertex S. The two triangles Δ PTU and Δ PRU lie in the same plane and are congruent, since they are the two triangles into which the parallelogram PTUR is separated by the diagonal $\overline{UP}$. Hence pyramids I and II have the same base area and the same altitude (the distance from S to plane PTUR), and so by Theorem 16-8 they have the same volume. In the same way, thinking of pyramids II and III as having bases SUR and SQR and common vertex P, we see that II and III have the same volume. Therefore the volume of all three pyramids is the same number, V, and the volume of the prism is 3V. If area $\Delta PQR = A$ and the altitude of SPQR = h, then

$$3V = Ah,$$

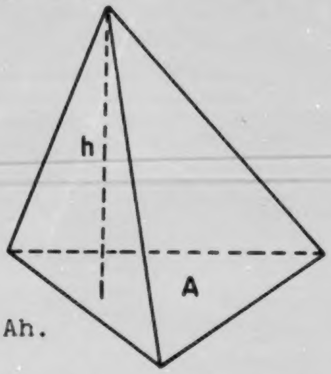
whence
$$V = \frac{1}{3} Ah$$ which was to be proved.

The same result holds for pyramids in general:

Theorem 16-10. The volume of a pyramid is one-third the product of its altitude and its base area.



$$V = \frac{1}{3} Ah.$$



Proof: Given a pyramid of altitude h and base area A. Take a triangular pyramid of the same altitude and base area, with its base in the same plane. By the Pyramid Cross-Section Theorem, cross-sections at the same level have the same area. Therefore, by Cavalieri's Principle, the two pyramids have the same volume. Therefore the volume of each of them is $\frac{1}{3}$ Ah, which was to be proved.

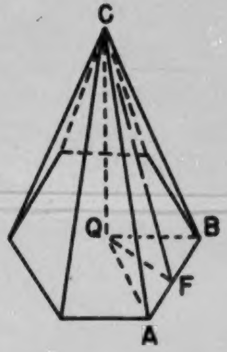
Problem Set 16-3

1. A rectangular tank 5' × 4' is filled with water to a depth of 9". How many cubic feet of water are in the tank? How many gallons? (1 gallon = 231 cubic inches.)

2. A lump of metal submerged in a rectangular tank of water 20 inches long and 8 inches wide raises the level of the water 4.6 inches. What is the volume of the metal?

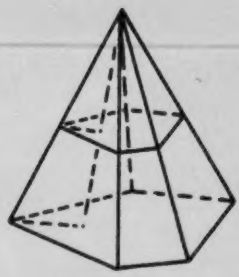
3. If one fish requires a gallon of water for good health, how many fish can be kept in an aquarium 2 feet long, $1\frac{1}{2}$ feet wide, and $1\frac{1}{2}$ feet deep?

4. If one edge of the base of a regular hexagonal pyramid is 12 inches and the altitude of the pyramid is 9 inches, what is the lateral area? What is the volume?

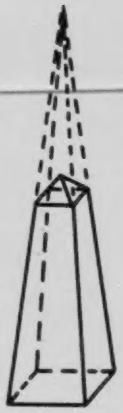


5. The volume of a pyramidal tent with a square base is 1836 cubic feet. If the side of the base is 18 feet, find the height of the tent.

6. A plane bisects the altitude of a pyramid and is parallel to its base. What is the ratio of the volumes of the solids above and below the plane?



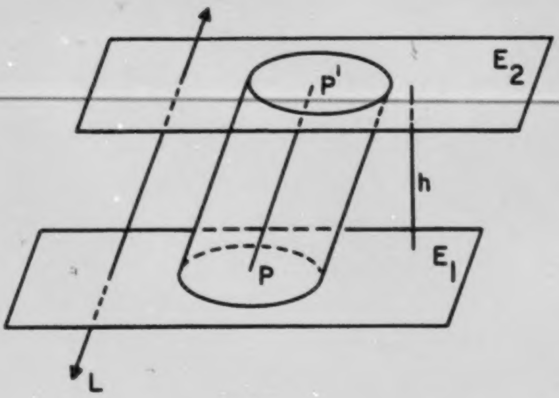
*7. A monument has the shape of an obelisk -- a square pyramid cut off at a certain height and capped with a second square pyramid. The vertex of the small pyramid is 2 feet above its base and 32 feet above the ground. If the base pyramid had been continued to its vertex it would have been 60 feet tall. Find the volume of the obelisk if each side of the base, at the ground, is 4 feet long.



*8. State and illustrate a principle, corresponding to Cavalieri's Principle, having the conclusion that two plane regions have equal areas.

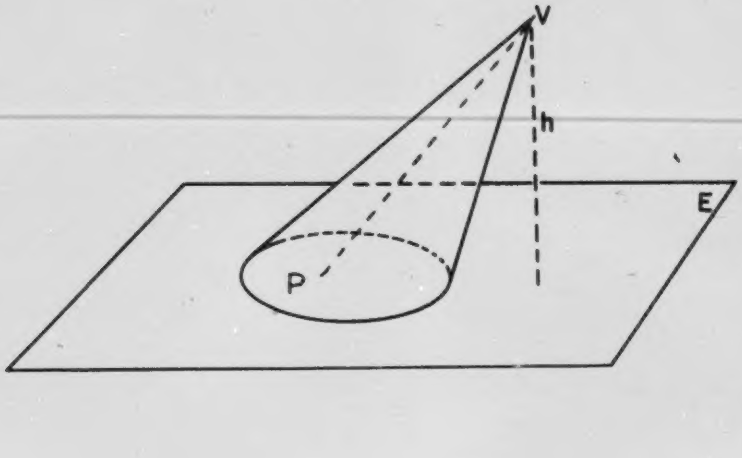
16-4. Cylinders and Cones.

Note that in the definition of a prism, and of associated terms in Section 16-1, it is not necessary to restrict K to be a polygonal region. K could in fact be any point set in E_1. Such tremendous generality is not needed, but we certainly can consider the case in which K is a circular region, the union of a circle and its interior. In this case we call the resulting solid a circular cylinder. You should write out a definition of a circular cylinder for yourself. You can use the following figure to help you.



We can have cylinders with other kinds of bases, such as elliptic cylinders, but the circular cylinder is by far the most common and the only one considered in elementary geometry.

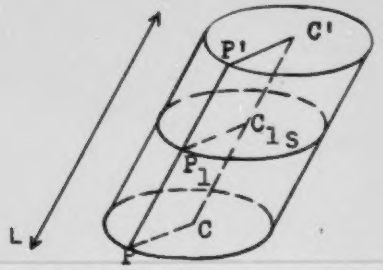
Just as the definition of a circular cylinder is analogous to that of a prism, the definition of a circular cone is analogous to the definition of a pyramid. Check your understanding of this by writing out a definition of a circular cone. You can use the notation of the following figure to help you.



Definition: If the center of the base circle is the foot of the perpendicular from V to E, the cone is called a right circular cone.

The following analogs of the theorems on prisms and pyramids are provable by the same general methods. We omit the details.

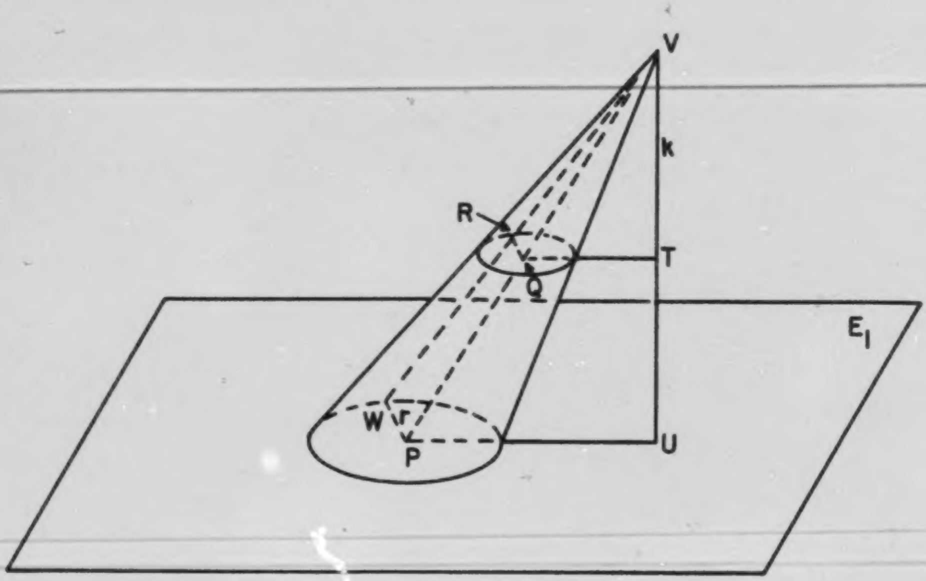
Theorem 16-11. A cross-section of a circular cylinder is a circular region congruent to the base.



Idea of proof: Let C be the center and r the radius of the base. Then, by parallelograms, $P_1C_1 = PC = r$.

Theorem 16-12. The area of a cross-section of a circular cylinder is equal to the area of the base.

Theorem 16-13. A cross-section of a cone of altitude h, made by a plane at a distance k from the vertex, is a circular region whose area has a ratio to the area of the base of $(\frac{k}{h})^2$.



Idea of proof: Let $VU = h$.

(1) $\Delta\ VQT \sim \Delta\ VPU$.

$$\frac{VQ}{VP} = \frac{k}{h}.$$

(2) $\Delta\ VQR \sim \Delta\ VPW$.

$$\frac{QR}{PW} = \frac{k}{h}.$$

(3) $QR = \frac{k}{h} PW$.

Since PW has a constant value, regardless of the position of W, then QR has a constant value. Thus, all points R lie on a circle. The corresponding circular region is the cross-section.

(4) $\dfrac{\text{area of circle with center } Q}{\text{area of circle with center } P} = \left(\frac{k}{h}\right)^2$.

We can now use Cavalieri's Principle to find the volumes of cylinders and cones.

Theorem 16-14. The volume of a circular cylinder is the product of the altitude and the area of the base.

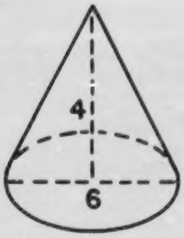
Proof like that of Theorem 16-7.

Theorem 16-15. The volume of a circular cone is one-third the product of the altitude and the area of the base.

Proof like that of Theorem 16-10.

Problem Set 16-4

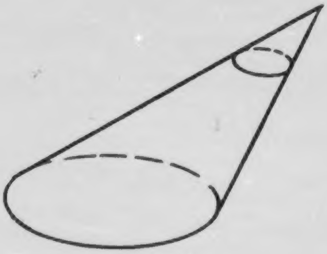
1. Find the volume of this right circular cone.



2. Find the number of gallons of water which a conical tank will hold if it is 30 inches deep and the radius of the circular top is 14 inches. (There are 231 cubic inches in a gallon. Use $\frac{22}{7}$ as an approximation of π. Why is $\frac{22}{7}$ a more convenient approximation than 3.14 in problems containing the number 231?)

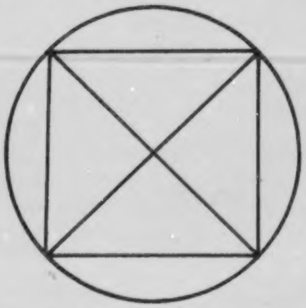
3. A drainage tile is a cylindrical shell 16 inches long. The inside and outside diameters are 5 inches, and 5.6 inches. Find the volume of clay necessary to make the tile.

4. A certain cone has a volume of 27 cubic inches. Its height is 5 inches. A second cone is cut from the first by a plane parallel to the base and two inches below the vertex. Find the volume of the second cone.



5. On a shelf in the supermarket stand two cans of imported olives. The first is twice as tall as the second, but the second has a diameter twice that of the first. If the second costs twice as much as the first, which is the better buy?

6. In this figure we are looking down upon a pyramid, whose base is a square, inscribed in a right circular cone. If the altitude of the cone or pyramid is 36 and a base edge of the pyramid is 20, find the volume of each.



7. Figure 1 represents a cone in a cylinder and Figure 2, two congruent cones in a cylinder. If the cylinders are the same size, compare the volume of the cone in Figure 1 with the volume of the two cones in Figure 2. Would your conclusion be changed if the cones in Figure 2 were not congruent?

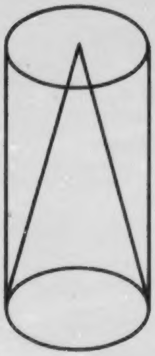


Fig. 1.

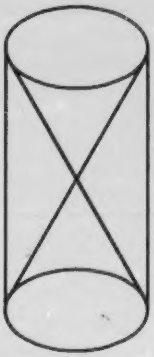
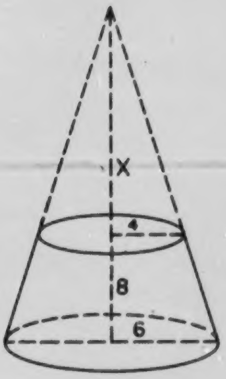


Fig. 2.

8. A right circular cone stands inside a right circular cylinder of same base and height. Write a formula for the volume of the space between the cylinder and the cone.

*9. If a plane parallel to the base of a cone (or pyramid) cuts off another cone (or pyramid) then the solid between the parallel plane and the base is called a frustum. A frustum of a cone has a lower radius of 6 inches, an upper radius of 4 inches and a height of 8 inches. Find its volume.

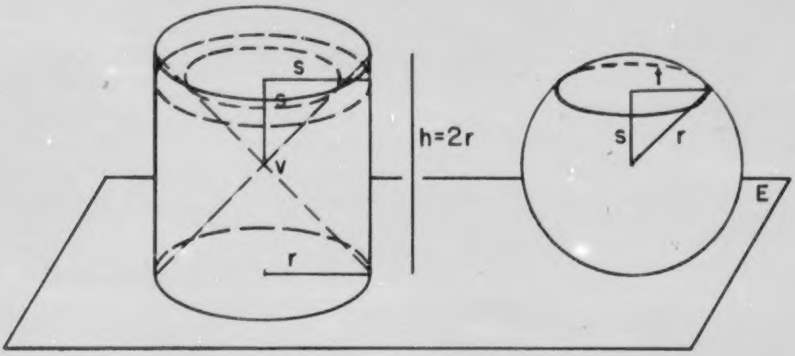


16-5. Spheres; Volume and Area.

By the volume of a sphere we mean the volume of the solid which is the union of the sphere and its interior.

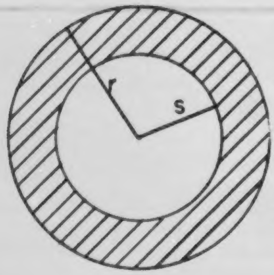
Theorem 16-16. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

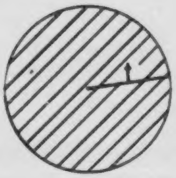
Proof: Given a sphere of radius r, let E be a tangent plane. In E take a circle of radius r and consider a right cylinder with this circle as base, altitude $2r$, and lying on the same side of E as the sphere.



Finally, consider two cones, with the two bases of the cylinder as their bases, and their common vertex V at the mid-point of the axis of the cylinder.

Take a cross-section of each solid by a plane parallel to E and at a distance s from V. The cross-sections will look like this:





The area of the section of the sphere is

$$A_1 = \pi t^2 = \pi(r^2 - s^2)$$

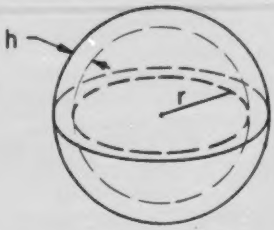
by the Pythagorean Theorem. We wish to compare this with the section of the solid lying between the cones and the cylinder, that is, outside the cones, but inside the cylinder. This section is a circular ring, whose outer radius is r and whose inner radius is s. (Why?) Hence, its area is

$$A_2 = \pi r^2 - \pi s^2 = \pi(r^2 - s^2).$$

Thus, $A_1 = A_2$, and by Cavalieri's Principle the volume of the sphere is equal to the volume between the cones and the cylinder. Therefore the volume of the sphere is the difference of the volume of the cylinder and twice the volume of one cone, that is,

$$\pi r^2 \cdot 2r - 2 \cdot \frac{1}{3}\pi r^2 \cdot r = \frac{4}{3}\pi r^3.$$

Using the formula for the volume of a sphere, we can get a formula for the area of the surface of a sphere. Given a sphere of radius r, form a slightly larger sphere, of radius $r + h$. The solid lying between the two spherical surfaces is called a spherical shell, and looks like this:



Let the surface area of the inner sphere be S. The volume V of the shell is then approximately hS. Thus, approximately, $S = \frac{V}{h}$. As the shell gets thinner, the approximation gets better and better. Thus, as h gets smaller and smaller, we have

$$\frac{V}{h} \longrightarrow S.$$

But we can calculate $\frac{V}{h}$ exactly, and see what it approaches when h becomes smaller and smaller. This will tell us what S is. The volume V is the difference of the volumes of the two spheres. Therefore:

$$\begin{aligned} V &= \tfrac{4}{3}\pi(r + h)^3 - \tfrac{4}{3}\pi r^3 \\ &= \tfrac{4}{3}\pi[(r + h)^3 - r^3] \\ &= \tfrac{4}{3}\pi[r^3 + 3r^2h + 3rh^2 + h^3 - r^3] \\ &= \tfrac{4}{3}\pi[3r^2h + 3rh^2 + h^3]. \end{aligned}$$

(You should check, by multiplication, that $(r + h)^3 = r^3 + 3r^3h + 3rh^2 + h^3$.)

Therefore
$$\begin{aligned} \frac{V}{h} &= \tfrac{4}{3}\pi[3r^2 + 3rh + h^2] \\ &= 4\pi r^2 + h[4\pi r + \tfrac{4}{3}\pi h]. \end{aligned}$$

Here the entire second term approaches zero, because $h \longrightarrow 0$. Therefore $\frac{V}{h} \longrightarrow 4\pi r^2$, and so $S = 4\pi r^2$. Thus we have the theorem:

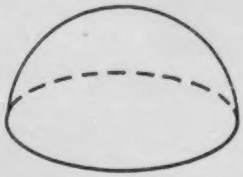
__Theorem__ __16-17__. The surface area of a sphere of radius r is

$$S = 4\pi r^2.$$

Thus we end this chapter with the interesting fact that the surface area of a sphere of radius r is $4\pi r^2$. Have you noticed that the surface area is exactly 4 times as great as the area of a great circle of the sphere?

__Problem__ __Set__ __16-5__

1. Compute the surface area and the volume of a sphere having diameter 8.
2. The radius of one sphere is twice as great as the radius of a second sphere. State a ratio expressing a comparison of their surface areas; their volumes. If the radius of one sphere is three times as great as the radius of another sphere, compare their surface areas; their volumes.
3. A spherical storage tank has a radius of 7 feet. How many gallons will it hold? (Use $\pi = \frac{22}{7}$.)
4. A large storage shed is in the shape of a hemisphere. The shed is to be painted. If the floor of the shed required 17 gallons of paint, how much paint will be needed to cover the exterior of the shed?

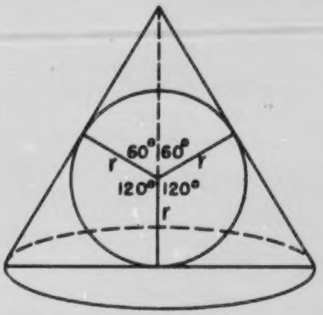


5. It was shown by Archimedes (287-212 B.C.) that the volume of a sphere is $\frac{2}{3}$ that of the smallest right circular cylinder which can contain it. Verify this.
6. An ice cream cone 5 inches deep and 2 inches in top diameter has placed on top of it two hemispherical scoops of ice cream also of 2 inch diameter. If the ice cream melts into the cone, will it overflow?

7. a. Show that if the length of a side of one cube is four times that of another cube the ratio of their volumes is 64 to 1.

   b. The moon has a diameter about $\frac{1}{4}$ that of the earth. How do their volumes compare?

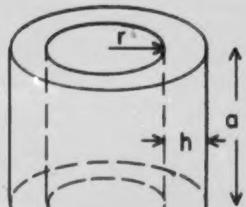
8. In the figure, the sphere, with radius r, is inscribed in the cone. The measure of the angles between the altitude and the radii to points of tangency are as shown. Find the volume of the cone in terms of r.



*9. The city engineer who was six feet tall walked up to inspect the new spherical water tank. When he had walked to a place 18 feet from the point where the tank rested on the ground he bumped his head on the tank. Knowing that the city used 10,000 gallons of water per hour, he immediately figured how many hours one tank full would last. How did he do it and what was his result?

*10. Half the air is let out of a rubber balloon. If it continues to be spherical in shape how does the resulting radius compare with the original radius?

*11. Use the method by which Theorem 16-17 was derived to show that the lateral area of a right circular cylinder is $2\pi ra$ where a is its altitude and r the radius of its base.



Review Problems

1. If the base of a pyramid is a region whose boundary is a rhombus with side 16 and an angle whose measure is 120, then

    a. any cross-section is a region whose boundary is a ______ and whose angles measure ______ and ______.

    b. the length of a side of a cross-section midway between the vertex and the base is _______.

    c. the area of a cross-section midway between the vertex and the base is ________.

2. A spherical ball of diameter 5 has a hollow center of diameter 2. Find the approximate volume of the shell.

3. Find the altitude of a cone whose radius is 5 and whose volume is 500.

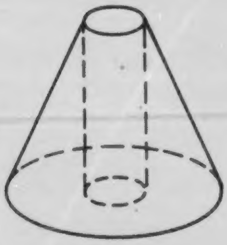
4. A pyramid has an altitude of 12 inches and volume of 432 cubic inches. What is the area of a cross-section 4 inches above the base?

5. Given two cones such that the altitude of the first is twice the altitude of the second and the radius of the base of the first is half the radius of the base of the second. How do the volumes compare?

6. A cylindrical can with radius 12 and height 20 is full of water. If a sphere of radius 10 is lowered into the can and then removed, what volume of water will remain in the can?

7. A sphere is inscribed in a right circular cylinder, so that it is tangent to both bases. What is the ratio of the volume of the sphere to the volume of the cylinder?

*8. The altitude of a right circular cone is 15 and the radius of its base is 8. A cylindrical hole of diameter 6 is drilled through the cone with the center of the drill following the axis of the cone, leaving a solid as shown in the figure. What is the volume of this solid?



9. Prove: If the base of a pyramid is a parallelogram region, the plane determined by the vertex of the pyramid and a diagonal of the base divides the pyramid into two pyramids of equal volume.

*10. Prove that a sphere can be circumscribed about a rectangular parallelepiped.

Chapter 17

PLANE COORDINATE GEOMETRY

17-1. Introduction.

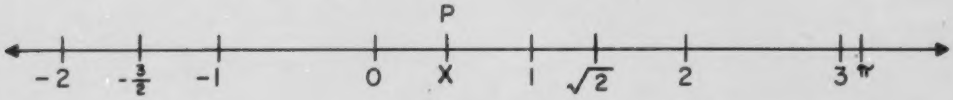
Mathematics is the only science in which practically nothing ever has to be thrown away. Of course, mathematicians are people, and being people, they make mistakes. But these mistakes usually get caught pretty quickly. Therefore, when one generation has learned something about mathematics, the next generation can go on to learn some more, without having to stop to correct serious errors in the work that was supposed to have been done already.

One symptom of this situation is the fact that nearly everything that you have been learning about geometry, so far in this course, was known to the ancient Greeks, over two thousand years ago.

The first really big step forward in geometry, after the Greeks, was in the seventeenth century. This was the discovery of a new method, called *coordinate geometry*, by Rene Descartes (1596-1650). In this chapter we will give a short introduction to coordinate geometry -- just about enough to give you an idea of what it is like and how it works.

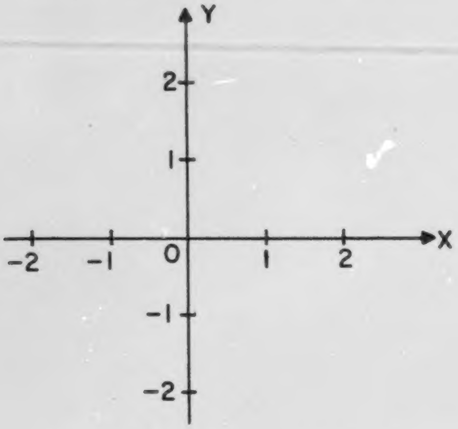
17-2. Coordinate Systems in a Plane.

In Chapter 2 we learned how to set up coordinate systems on a line.



Once we have set up a coordinate system, every number describes a point, and every point P is determined when its coordinate x is named.

In coordinate geometry, we do the same sort of thing in a plane, except that in a plane a point is described not by a single number, but by a pair of numbers. The scheme works like this:

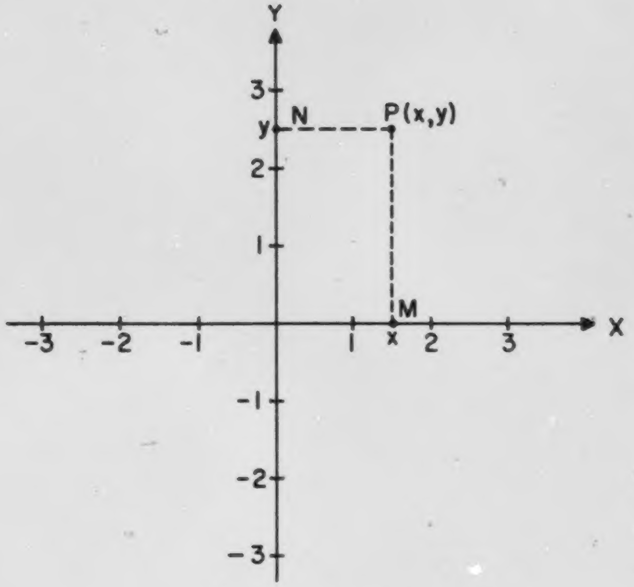


First we take a line X in the plane, and set up a coordinate system on X. This line will be called the x-axis. In a figure we usually use an arrow-head to emphasize the positive direction on the x-axis.

Next we let Y be the perpendicular to the x-axis through the point O whose coordinate is zero, and we set up a coordinate system on Y. By the Ruler Placement Postulate this can be done so that point O also has coordinate zero on Y. Y will be called the y-axis. As before, we indicate the positive direction by an arrow-head. The intersection O of the two axes is called the origin.

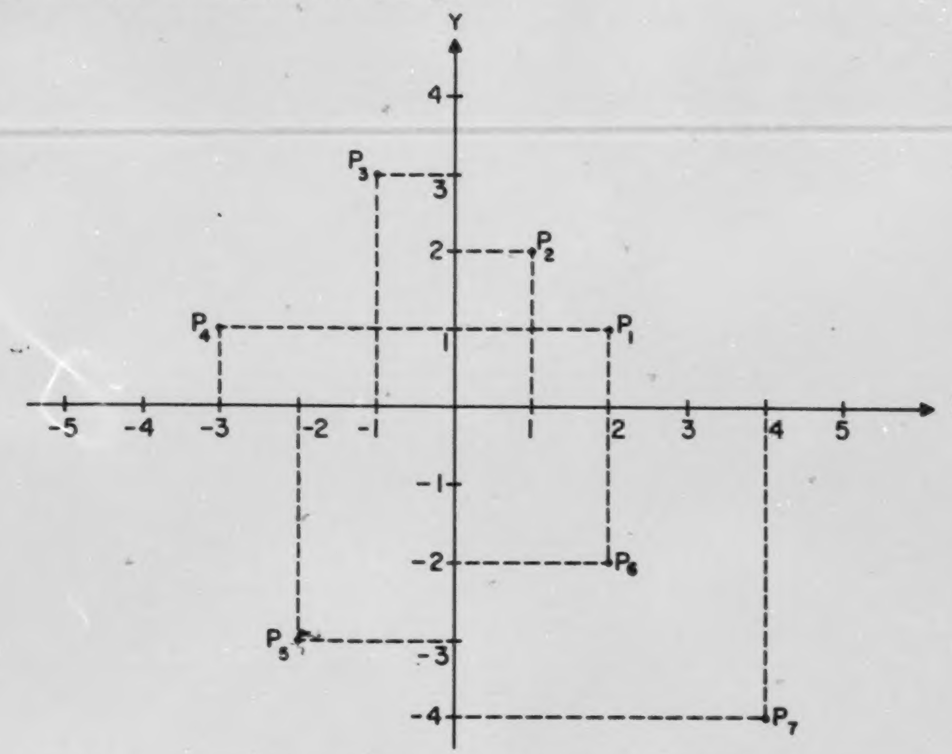
We can now describe any point in the plane by a pair of numbers. The scheme is this. Given a point P, we drop a perpendicular to the x-axis, ending at a point M, with coordinate x. We drop a perpendicular to the y-axis, ending at a point N, with coordinate y. (In accord with Section 10-3 we can call M and N the projections of P into X and Y.)

Definitions: The numbers x and y are called the coordinates of the point P; x is the x-coordinate and y is the y-coordinate.



In the figure $x = 1\frac{1}{2}$ and $y = 2\frac{1}{2}$. The point P therefore has coordinates $1\frac{1}{2}$ and $2\frac{1}{2}$. We write these coordinates in the form $(1\frac{1}{2}, 2\frac{1}{2})$, giving the x-coordinate first. To indicate that point P has these coordinates we write $P(1\frac{1}{2}, 2\frac{1}{2})$ or $P:(1\frac{1}{2}, 2\frac{1}{2})$.

Let us look at some more examples.



We read off the coordinates of the points by following the dotted lines. Thus the coordinates, in each case, are as follows:

$P_1(2,1)$

$P_2(1,2)$

$P_3(-1,3)$

$P_4(-3,1)$

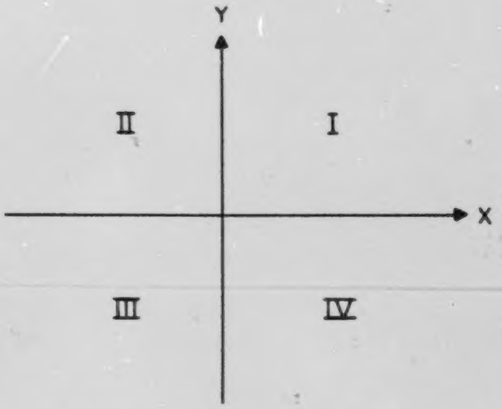
$P_5(-2,-3)$

$P_6(2,-2)$

$P_7(4,-4)$

Notice that the order in which the coordinates are written makes a difference. The point with coordinates (2,1) is not the same as point (1,2). Thus, the coordinates of a point are really an ordered pair of real numbers, and you can't tell where the point is unless you know the order in which the coordinates are given. The convention of having the first number of the ordered pair be the x-coordinate, and the second the y-coordinate, is highly important.

Just as a single line separates the plane into two parts (called half-planes) so the two axes separate the plane into four parts, called quadrants. The quadrants are identified by number, like this:

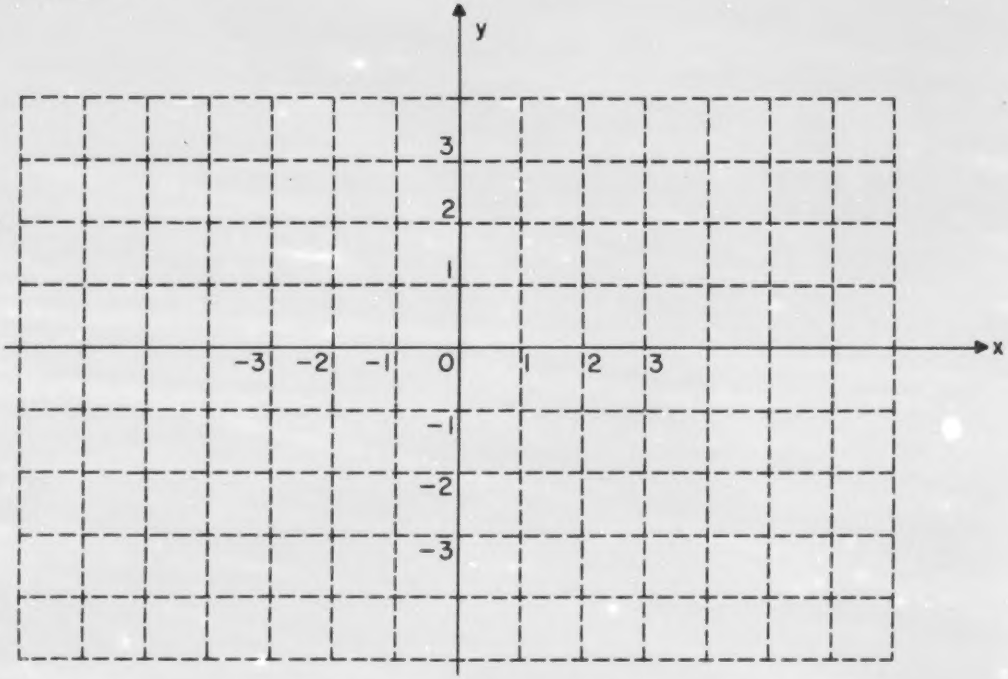


We have shown that any point of our plane determines an ordered pair of numbers. Can we reverse the process? That is, given a pair of numbers (a,b) can we find a point whose coordinates are (a,b)? The answer is easily seen to be "yes". In fact, there is exactly one such point, obtained as the intersection of the line perpendicular to the x-axis at the point whose coordinate is a and the line perpendicular to the y-axis at the point whose coordinate is b.

Thus, we have a one-to-one correspondence between points in the plane and ordered pairs of numbers. Such a correspondence is called a coordinate system in the plane. A coordinate system is specified by choosing a measure of distance, an x-axis, a y-axis perpendicular to it and a positive direction on each. As long as we stick to a specific coordinate system, which will be the case in all our problems in this book, each point P is associated with exactly one number pair (a,b), and each number pair with exactly one point. Hence, it will cause no confusion if we say the number pair is the point, thus enabling us to use such convenient phrases as "the point (2,3)" or "P = (a,b)".

17-3. How to Plot Points on Graph Paper.

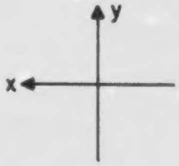
As a matter of convenience, we ordinarily use printed graph paper for drawing figures in coordinate geometry. The horizontal and vertical lines are printed; we have to draw everything else for ourselves.

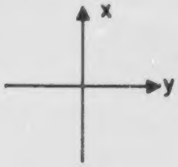


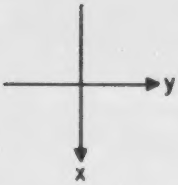
In the figure above, the dotted lines represent the lines that are already printed on the paper. The x-axis and the y-axis should be drawn with a pen or a pencil. Notice that the x-axis is labeled x rather than X; this is customary. Here the symbol x is not the name of anything, but merely a reminder that the coordinates on this axis are going to be denoted by the letter x. Similarly, for the y-axis. Next, the points with coordinates (1,0) and (0,1) must be labeled in order to indicate the unit to be used.

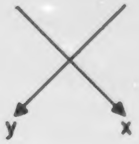
This is the usual way of preparing graph paper for plotting points. We could have indicated a little less or a lot more. For your own convenience, it is a good idea to show more than this. But if you show less, then your work may be actually unintelligible.

Note that we could draw the axes in any of the following positions:









and so on. There is nothing logically wrong with any of these ways of drawing the axes. People find it easier to read each other's graphs, however, if they agree at the outset that the x-axis is to be horizontal, with coordinates increasing from left to right, and the y-axis is to be vertical, with coordinates increasing from bottom to top.

Problem Set 17-3

1. Suggest why the kind of coordinate system used in this chapter is sometimes called "Cartesian".

2. What are the coordinates of the origin?

3. What is the y-coordinate of the point (7,-3)?

4. Name the point which is the projection of (0,-4) into the x-axis.

5. Which pair of points are closer together, (2,1) and (1,2) or (2,1) and (2,0)?

6. In which quadrant is each of the following points?

    a. (5,-3).

    b. (-5,3).

    c. (5,3).

    d. (-5,-3).

7. What are the coordinates of a point which does not lie in any quadrant?

8. The following points are projected into the x-axis. Write them in such an order that their projections will be in order from left to right.

    A:(6,-3). B:(-2,5). C:(0,-4). D:(-5,0).

9. If the points in the previous problem are projected into the y-axis arrange them so their projections will be in order from bottom to top.

10. If s is a negative number and r a positive number, in what quadrant will each of the following points lie?

    a. (s,r).

    b. $(-s,r)$.

    c. $(-s,-r)$.

    d. $(s,-r)$.

    e. (r,s).

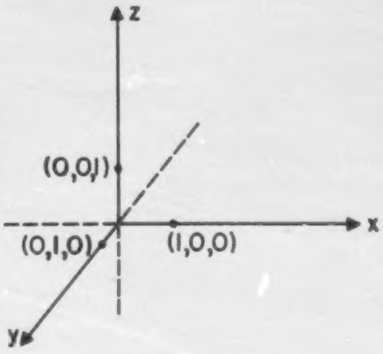
    f. $(r,-s)$.

    g. $(-r,-s)$.

    h. $(-r,s)$.

11. Set up a coordinate system on graph paper. Using segments draw some simple picture on the paper. On a separate paper list in pairs the coordinates of the end points of the segments in your picture. Exchange your list of coordinates with another student, and reproduce the picture suggested by his list of coordinates.

*12. A three dimensional coordinate system can be formed by considering three mutually perpendicular axes as shown. The y-axis, while drawn on this paper, represents a line perpendicular to the plane of the paper. The negative portions of the x, y and z axes extend to the left, to the rear, and down respectively. Taken in pairs the three axes determine three planes called the yz-plane, the xz-plane, and the xy-plane. A point (x,y,z) is located by its three coordinates: the x-coordinate is the coordinate of its projection into the x-axis; the y and z coordinates are defined in a corresponding manner.



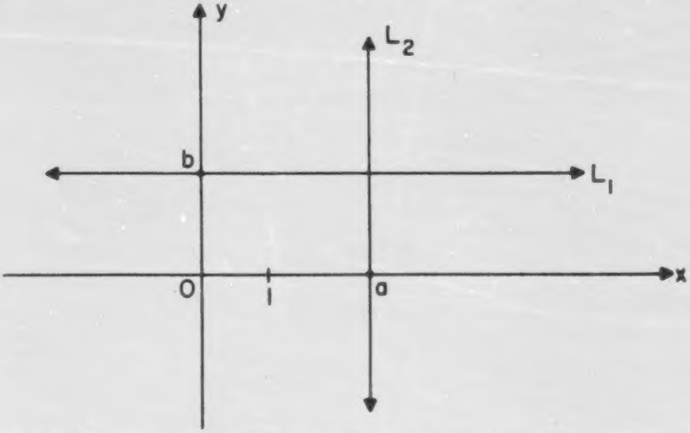
a. On which axis will each of these points lie?
$(0,5,0)$; $(-1,0,0)$; $(0,0,8)$.

b. On which plane will each of these points lie?
$(2,0,3)$; $(0,5,-7)$; $(1,1,0)$.

c. What is the distance of the point $(3,-2,4)$ from the xy-plane? from the xz-plane? from the yz-plane?

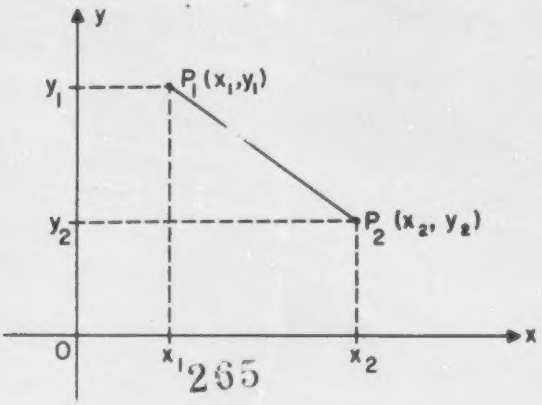
17-4. The Slope of a Non-Vertical Line.

The x-axis, and all lines parallel to it, are called horizontal. The y-axis, and all lines parallel to it, are called vertical. Notice that these terms are defined in terms of the coordinate system that we have set up.



On the horizontal line L_1, all points have the same y-coordinate b, because the point (0,b) on the y-axis is the foot of all the perpendiculars from points of L_1. For the same sort of reason, all points of the vertical line L_2 have the same x-coordinate a. Of course, a segment is horizontal (or vertical) if the line containing it is horizontal (or vertical).

Consider now a segment $\overline{P_1P_2}$, where $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$, and suppose that $\overline{P_1P_2}$ is not vertical.



Definition: The slope of $\overline{P_1P_2}$ is the number $m = \frac{y_2 - y_1}{x_2 - x_1}$.

This really is a number: since the segment is not vertical, P_1 and P_2 have different x-coordinates, and so the denominator is not zero. Some things about the slope are easy to see.

(1) It is important that the order of naming the coordinates is the same in the numerator as in the denominator. Thus, if we wish to find the slope of $\overline{PQ}$, where $P = (1,3)$ and $Q = (4,2)$ we can either choose $P_1 = P$, $x_1 = 1$, $y_1 = 3$, $P_2 = Q$, $x_2 = 4$, $y_2 = 2$, giving slope of $\overline{PQ} = \frac{2 - 3}{4 - 1} = -\frac{1}{3}$;

or $P_1 = Q$, $x_1 = 4$, $y_1 = 2$, $P_2 = P$, $x_2 = 1$, $y_2 = 3$,

giving slope of $\overline{PQ} = \frac{3 - 2}{1 - 4} = -\frac{1}{3}$.

What we cannot say is

$$\text{slope of } \overline{PQ} = \frac{3 - 2}{4 - 1} \text{ or } \frac{2 - 3}{1 - 4}.$$

Notice that if the points are named in reverse order, the slope is the same as before. Algebraically,

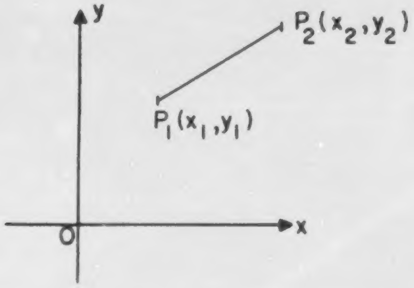
$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

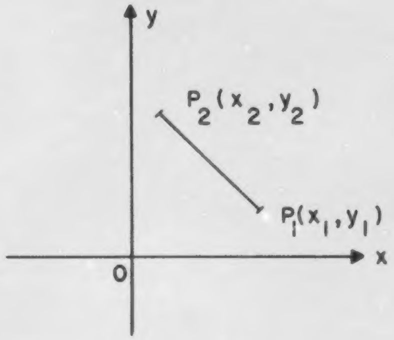
Thus the value of m depends only on the segment, not on the order in which the end-points are named.

(2) If $m = 0$, then the segment is horizontal. (Algebraically, a fraction is zero only if its numerator is zero, and this means that $y_2 = y_1$.)

(3) If the segment slopes upward from left to right, as in the left hand figure on page 578, then $m > 0$, because the numerator and denominator are both positive (or both negative, if we reverse the order of the end-points.)

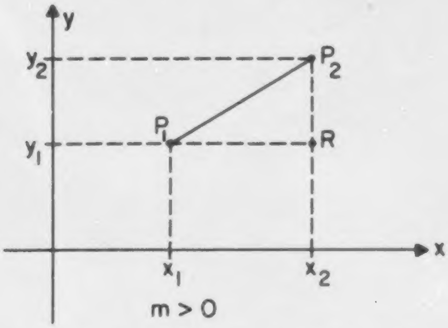
(4) If the segment slopes upward from right to left as in the right hand figure below, then $m < 0$. This is because m can be written as a fraction with a positive numerator $y_2 - y_1$ and a negative denominator $x_2 - x_1$ (or equivalently, a negative numerator $y_1 - y_2$ and a positive denominator $x_1 - x_2$).


y
$P_2(x_2, y_2)$
$P_1(x_1, y_1)$
0
x

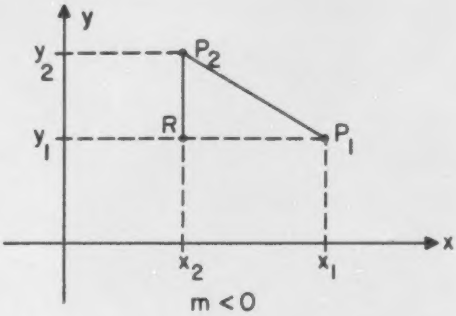

y
$P_2(x_2, y_2)$
$P_1(x_1, y_1)$
0
x

(5) We do not try to write the slope of a vertical segment, because the denominator would be zero, and so the fraction would be meaningless.

In either of the two figures above, we can complete a right triangle $\Delta\ P_1P_2R$, by drawing horizontal and vertical lines through P_1 and P_2, like this:


y
y_2
P_2
P_1
R
y_1
x_1
x_2
x

m > 0


y
y_2
P_2
R
P_1
y_1
x_2
x_1
x

m < 0

Since opposite sides of a rectangle are congruent, it is easy to see that

(1) if $m > 0$, then $m = \frac{RP_2}{P_1R}$ and

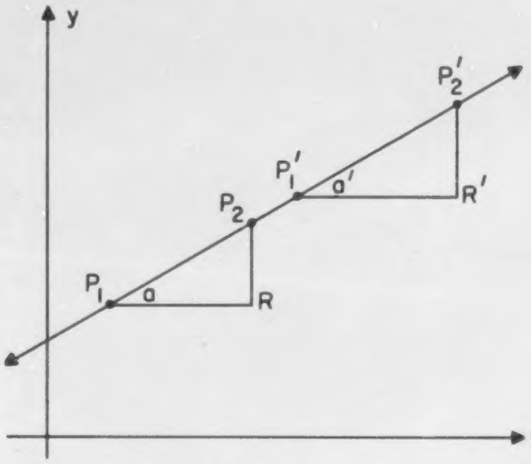
(2) if $m < 0$, then $m = -\frac{RP_2}{P_1R}$.

Once we know this much about slopes, it is easy to get our first basic theorem.

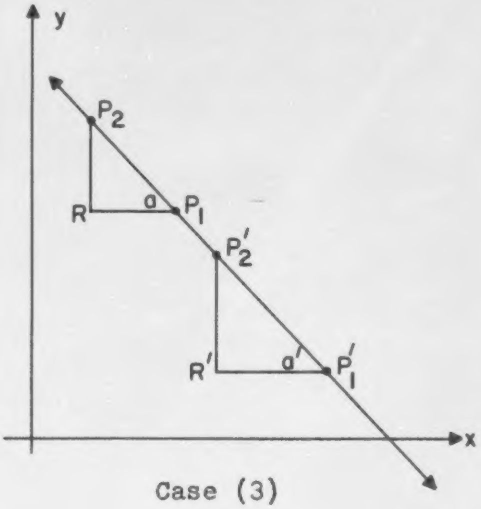
Theorem 17-1. On a non-vertical line, all segments have the same slope.

Proof: There are three cases to be considered.

Case (1): If the line is horizontal all segments on it have slope zero.



Case (2)



Case (3)

In either of the other cases illustrated above, $\angle a \cong \angle a'$, and since the triangles are right triangles, this means that

$$\Delta\ P_1P_2R \sim \Delta\ P_1'P_2'R'.$$

Therefore, in either case,

$$\frac{RP_2}{P_1R} = \frac{R'P_2'}{P_1'R'}.$$

In Case (2), these fractions are the slopes of $\overline{P_1P_2}$ and $\overline{P_1'P_2'}$ and therefore the segments have the same slope. In Case (3), the slopes are the negatives of the same fractions, and are therefore equal.

Theorem 17-1 means that we can talk not only about the slopes of segments but also about the slopes of lines: the slope of a non-vertical line is the number m which is the slope of every segment of the line.

Problem Set 17-4

1. Replace the "?" in such a way that the line through the two points will be horizontal.
   a. $(5,7)$ and $(-3,?)$.
   b. $(0,-1)$ and $(4,?)$.
   c. (x_1,y_1) and $(x_2,?)$.
2. Replace the "?" in such a way that the line through the two points will be vertical.
   a. $(?,2)$ and $(6,-4)$.
   b. $(-3,-1)$ and $(?,0)$.
   c. (x_1,y_1) and $(?,y_2)$.
3. By visualizing the points on a coordinate system in parts (a), (b), and (c), give the distance between:
   a. $(5,0)$ and $(7,0)$.
   b. $(5,1)$ and $(7,1)$.
   c. $(-3,-4)$ and $(-6,-4)$.
   d. What is alike about parts (a), (b) and (c)?

e. State a rule giving an easy method for finding the distance between such pairs of points.

f. Does your rule apply to the distance between (6,5) and (3,-5)?

4. By visualizing the points named in parts (a), (b), (c) and (d) on a coordinate system, give the distance between the points in each part.

a. (7,-3) and (7,0).

b. (-3,1) and (-3,-1).

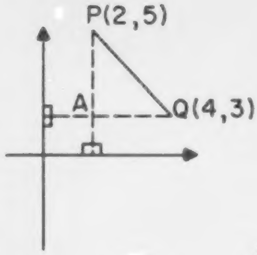
c. (6,8) and (6,4).

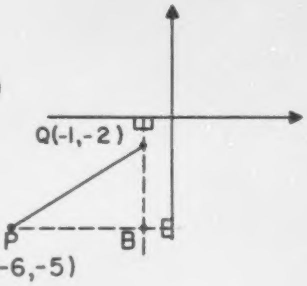
d. (x_1,y_1) and (x_1,y_2).

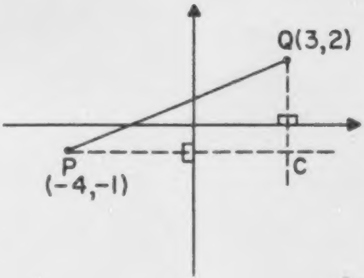
e. What is alike about parts (a), (b), (c) and (d)?

f. State a rule giving an easy method for finding the distance between such pairs of points.

5. With perpendiculars drawn as shown below, what are the coordinates of A, B and C?







6. Determine the distances from P and Q to points A, B and C in Problem 5.

7. Compute the slope of $\overline{PQ}$ for each figure in Problem 5.

8. A road goes up 2 feet for every 30 feet of horizontal distance. What is its slope?

9. Determine the slope of the segment joining each of the following point pairs.
   a. $(0,0)$ and $(6,2)$.
   b. $(0,0)$ and $(2,-6)$.
   c. $(3,5)$ and $(7,12)$.
   d. $(0,0)$ and $(-4,-3)$.
   e. $(-5,7)$ and $(3,-8)$.
   f. $(\frac{1}{2},\frac{1}{3})$ and $(\frac{1}{4},\frac{1}{5})$.
   g. $(-2.8,3.1)$ and $(2.2,-1.9)$.
   h. $(\frac{1}{240},0)$ and $(0,\frac{1}{80})$.

10. Replace the "?" by a number so that the line through the two points will have the slope given. (Hint: Substitute in the slope formula.)
   a. $(5,2)$ and $(?,6)$. $m = 4$.
   b. $(-3,1)$ and $(4,?)$. $m = \frac{1}{2}$.

*11. $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are non-vertical lines. Prove that $\overleftrightarrow{PA} = \overleftrightarrow{PB}$ if and only if they have the same slope; and consequently if $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ have different slopes, then P, A and B, cannot be collinear.

12. a. Is the point $B(4,13)$ on the line joining $A(1,1)$ to $C(5,17)$? (Hint: is the slope of $\overline{AB}$ the same as that of $\overline{BC}$?)
   b. Is the point $(2,-1)$ on the segment joining $(-5,4)$ to $(6,-8)$?

13. Determine the slope of a segment joining:
    a. $(0,n)$ and $(n,0)$.
    b. $(2d,-2d)$ and $(0,d)$.
    c. $(a+b,a)$ and $(a-b,b)$.

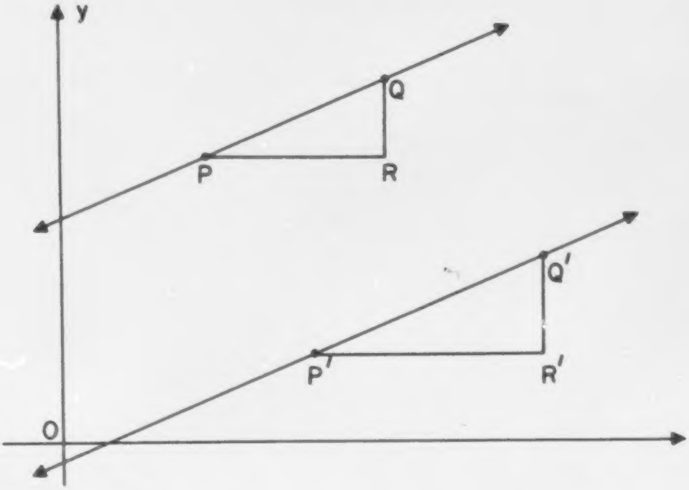
14. Given $A:(101,102)$, $B:(5,6)$, $C:(-95,-94)$, determine whether or not lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ coincide.

15. Given $A:(101,102)$, $B:(5,6)$, $C:(202,203)$, $D:(203,204)$. Are $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ parallel? Could they possibly coincide?

16. Draw the part of the first quadrant of a coordinate system having coordinates less than or equal to 5. Draw a segment through the origin which, if extended, would pass through $P(80000000,60000000)$.

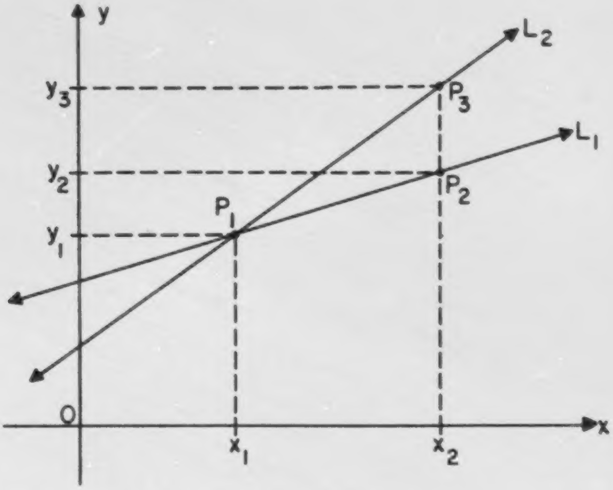
17-5. Parallel and Perpendicular Lines.

It is easy to see the algebraic condition for two non-vertical lines to be parallel.



If the lines are parallel, then $\Delta PQR \sim \Delta P'Q'R'$, and it follows, as in the proof of the preceding theorem, that they have the same slope.

Conversely, if two different lines have the same slope, then they are parallel. We prove this by the method of contradiction.

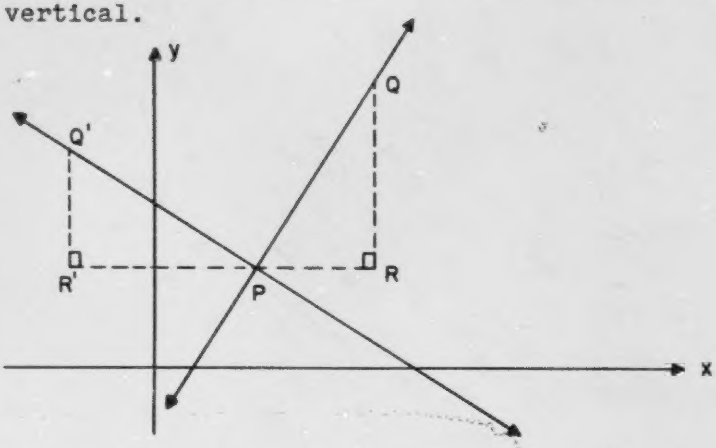


Assume as in the figure above that L_1 and L_2 are not parallel. If as shown in the figure P_1 is their point of intersection, and P_2 and P_3 have the same x-coordinate x_2, the slope of L_1 is $m_1 = \frac{y_2 - y_1}{x_2 - x_1}$, and the slope of L_2 is $m_2 = \frac{y_3 - y_1}{x_2 - x_1}$. Since $y_3 \neq y_2$, the fractions cannot be equal, and hence $m_1 \neq m_2$. Thus our initial assumption that the two lines L_1 and L_2 were not parallel has led us to a contradiction of the hypothesis that $m_1 = m_2$. Hence the two lines L_1 and L_2 must be parallel.

Thus we have the theorem:

Theorem 17-2. Two non-vertical lines are parallel if and only if they have the same slope.

Now turning to the condition for two lines to be perpendicular, let us suppose that we have given two perpendicular lines, neither of which is vertical.



Let P be their point of intersection. As in the figure, let Q be a point of one of the lines, lying above and to the right of P. And let Q' be a point of the other line, lying above and to the left of P, such that PQ' = PQ. We complete the right triangles Δ PQR and Δ Q'PR' as indicated in the figure. Then

$$\Delta PQR \cong \Delta Q'PR'. \qquad \text{(Why?)}$$

Therefore $$Q'R' = PR \text{ and } R'P = RQ.$$

and hence $$\frac{Q'R'}{R'P} = \frac{PR}{RQ}.$$

Let m be the slope of $\overleftrightarrow{PQ}$, and let m' be the slope of $\overleftrightarrow{PQ'}$.

Then $$m = \frac{RQ}{PR},$$

and $$m' = -\frac{Q'R'}{R'P} = -\frac{PR}{RQ}.$$

Therefore $$m' = -\frac{1}{m}.$$

That is, the slopes of perpendicular lines are the negative reciprocals of each other.

Suppose, conversely, that we know that $m' = -\frac{1}{m}$. We then construct $\Delta\ PQR$ as before, and we construct the right triangle $\Delta\ Q'PR'$ making $R'P = RQ$. We can then prove that $Q'R' = PR$; this gives the same congruence, $\Delta\ PQR \cong \Delta\ Q'PR'$, as before, and it follows that $\angle\ Q'PQ$ is a right angle and hence $\overleftrightarrow{PQ} \perp \overleftrightarrow{PQ'}$.

These two facts are stated together in the following theorem:

Theorem 17-3. Two non-vertical lines are perpendicular if and only if their slopes are the negative reciprocals of each other.

Notice that while Theorems 17-2 and 17-3 tell us nothing about vertical lines, they don't really need to, because the whole problem of parallelism and perpendicularity is trivial when one of the lines is vertical. If L is vertical, then L' is parallel to L if and only if L' is also vertical (and different from L.) And if L is vertical, then L' is perpendicular to L if and only if L' is horizontal.

Problem Set 17-5

1. Four points taken in pairs determine six segments. Which pairs of segments determined by the following four points are parallel? $A(3,6)$; $B(5,9)$; $C(8,2)$; $D(6,-1)$. (Caution: Two segments are not necessarily parallel if they have the same slope!)
2. Show by considering slopes that a parallelogram is formed by drawing segments joining in order $A(-1,5)$, $B(5,1)$, $C(6,-2)$ and $D(0,2)$.
3. Lines L_1, L_2, L_3 and L_4 have slopes $\frac{2}{3}$, -4, $-1\frac{1}{2}$ and $\frac{1}{4}$ respectively. Which pairs of lines are perpendicular?

4. It is asserted that both of the quadrilaterals whose vertices are given below are parallelograms. Without plotting the points, determine whether or not this is true.

   (1) A:(-5,-2), B:(-4,2), C:(4,6), D:(3,1).

   (2) P:(-2,-2), Q:(4,2), R:(9,1), S:(3,-3).

5. The vertices of a triangle are A(16,0), B(9,2) and C(0,0).

   a. What are the slopes of its sides?

   b. What are the slopes of its altitudes?

6. Show that the quadrilateral joining A(-2,2), B(2,-2), C(4,2), and D(2,4) is a trapezoid with perpendicular diagonals.

7. Show that a line through (3n,0) and (0,n) is parallel to a line through (6n,0) and (0,2n).

8. Show that a line through (0,0) and (a,b) is perpendicular to a line through (0,0) and (-b,a).

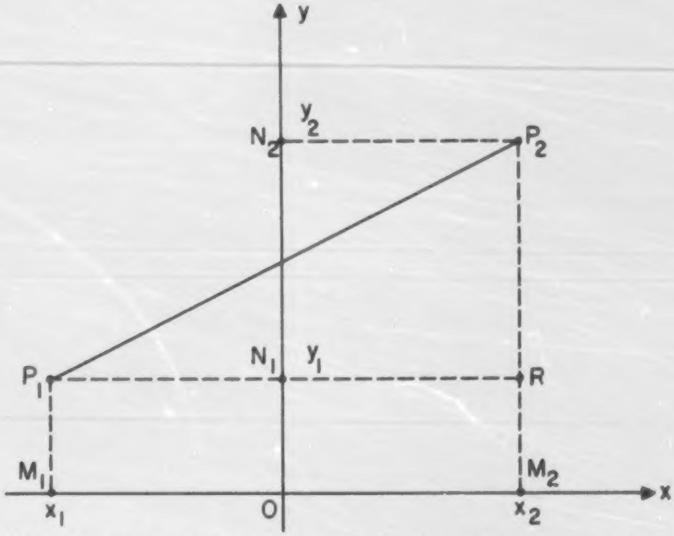
*9. Show that if a triangle has vertices $X(r,s)$, $Y(na+r,nb+s)$ and $Z(-mb+r,ma+s)$ it will have a right angle at X.

10. Given the points P(1,2), Q(5,-6) and R(b,b); determine the value of b so that $\angle PQR$ is a right angle.

11. $P = (a,1)$, $Q = (3,2)$, $R = (b,1)$, $S = (4,2)$. Prove that $\overleftrightarrow{PQ} \neq \overleftrightarrow{RS}$, and that if $\overline{PQ} \parallel \overline{RS}$ then $a = b - 1$.

17-6. The Distance Formula.

If we know the coordinates of two points P_1 and P_2 then we know where the points are, and so the distance P_1P_2 is determined. Let us now find out how the distance can be calculated. What we want is a formula that gives P_1P_2 in terms of the coordinates x_1, x_2, y_1 and y_2.



Let the projections M_1, M_2, N_1 and N_2 be as in the figure. By the Pythagorean Theorem, $(P_1P_2)^2 = (P_1R)^2 + (RP_2)^2$.

also $P_1R = M_1M_2$ and $RP_2 = N_1N_2$,

because opposite sides of a rectangle are congruent.

Therefore $(P_1P_2)^2 = (M_1M_2)^2 + (N_1N_2)^2$.

But we know that $M_1M_2 = |x_2 - x_1|$

and $N_1N_2 = |y_2 - y_1|$.

Therefore $(P_1P_2)^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$.

Of course, the square of the absolute value of a number is the same as the square of the number itself.

Therefore $(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$,

and since $P_1P_2 \geq 0$, this means that

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

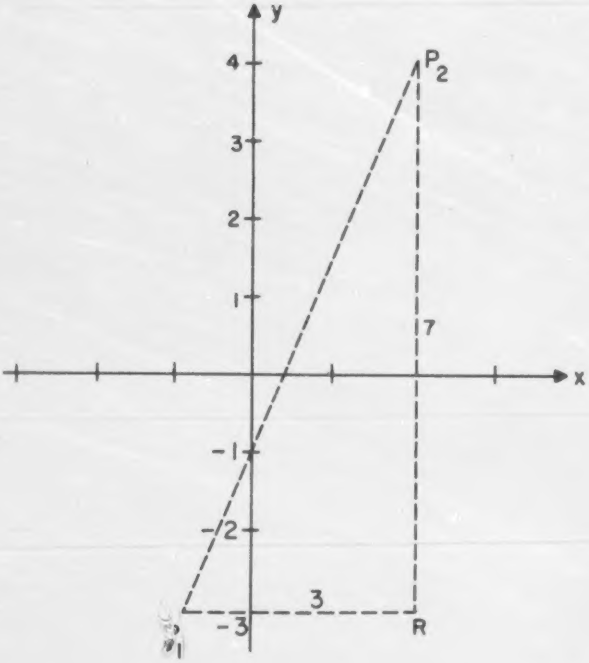
This is the formula that we are looking for. Thus we have the theorem:

Theorem 17-4. (The Distance Formula.) The distance between the points (x_1,y_1) and (x_2,y_2) is equal to

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

For example, take $P_1 = (-1,-3)$ and $P_2 = (2,4)$.

By formula,
$$\begin{aligned} P_1P_2 &= \sqrt{(2 + 1)^2 + (4 + 3)^2} \\ &= \sqrt{9 + 49} \\ &= \sqrt{58}. \end{aligned}$$



Of course, if we plot the points, as above, we can get the same answer directly from the Pythagorean Theorem; the legs of the right triangle ΔP_1RP_2 have lengths 3 and 7, so that $P_1P_2 = \sqrt{3^2 + 7^2}$, as before. If we find the distance this way, we are of course simply repeating the derivation of the distance formula in a specific case.

Problem Set 17-6

1. a. Without using the distance formula state the distance between each pair of the points: $A(0,3)$, $B(1,3)$, $C(-3,3)$ and $D(4.5,3)$.

   b. Without using the distance formula state the distance between each pair of the points: $A(2,0)$, $B(2,1)$, $C(2,-3)$ and $D(2,4.5)$.

2. a. Write a simple formula for the distance between (x_1,k) and (x_2,k). (Hint: The points would lie on a horizontal line.)

   b. Write a simple formula for the distance between (k,y_1) and (k,y_2).

3. Use the distance formula to find the distance between:

   a. $(0,0)$ and $(3,4)$.

   b. $(0,0)$ and $(3,-4)$.

   c. $(1,2)$ and $(6,14)$.

   d. $(8,11)$ and $(15,35)$.

   e. $(3,8)$ and $(-5,-7)$.

   f. $(-2,3)$ and $(-1,4)$.

   g. $(10,1)$ and $(49,81)$.

   h. $(-6,3)$ and $(4,-2)$.

4. a. Write a formula for the square of the distance between the points (x_1,y_1) and (x_2,y_2).

   b. Using coordinates write and simplify the statement: The square of the distance between $(0,0)$ and (x,y) is 25.

5. Show that the triangle with vertices $R(0,0)$, $S(3,4)$ and $T(-1,1)$ is isosceles by computing the lengths of its sides.

6. Using the converse of the Pythagorean Theorem show that the triangle joining $D(1,1)$, $E(3,0)$ and $F(4,7)$ is a right triangle with a right angle at D.

7. Given the points $A(-1,6)$, $B(1,4)$ and $C(7,-2)$. Prove, without plotting the points, that B is between A and C.

8. Suppose the streets in a city form congruent square blocks with avenues running east-west and streets north-south.

   a. If you follow the sidewalks, how far would you have to walk from the corner of 4th avenue and 8th street to the corner of 7th avenue and 12th street? (Use the length of 1 block as your unit of length.)

   b. What would be the distance "as the crow flies" between the same two corners?

9. Vertices W, X and Z of rectangle $WXYZ$ have coordinates $(0,0)$, $(a,0)$ and $(0,b)$ respectively.

   a. What are the coordinates of Y?

   b. Prove, using coordinates, that $WY = XZ$.

*10. a. Using 3-dimensional coordinates (see Problem 12 of Problem Set 17-3), compute the distance between $(0,0,0)$ and $(2,3,6)$.

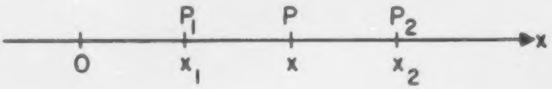
   b. Write a formula for the distance between $(0,0,0)$ and (x,y,z).

   c. Write a formula for the distance between $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

17-7. The Mid-Point Formula.

In Section 17-8 we will be proving geometric theorems by the use of coordinate systems. In some of these proofs, we will need to find the coordinates of the mid-point of a segment P_1P_2 in terms of the coordinates of P_1 and P_2.

First let us take the case where P_1 and P_2 are on the x-axis, with $x_1 < x_2$, like this:



and P is the mid-point, with coordinate x. Since $x_1 < x < x_2$, we know that $P_1P = x - x_1$ and $PP_2 = x_2 - x$.

Since P is the mid-point, this gives

$$x - x_1 = x_2 - x,$$

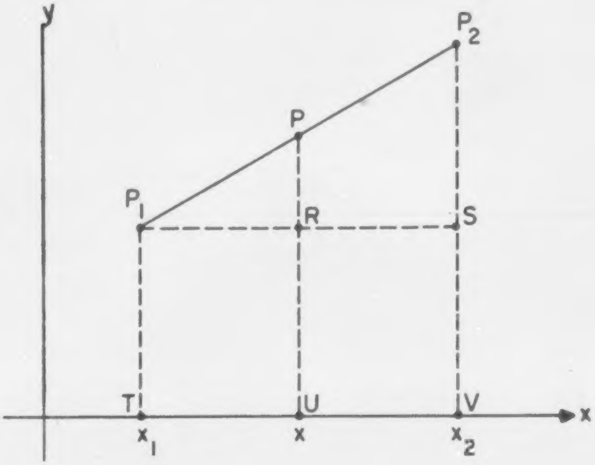
or

$$x = \frac{x_1 + x_2}{2}.$$

In the same way, on the y-axis,

$$y = \frac{y_1 + y_2}{2}.$$

Now we can handle the general case easily:



Since P is the mid-point of $\overline{P_1P_2}$, it follows by similar triangles that R is the mid-point of $\overline{P_1S}$. Since opposite sides of a rectangle are congruent, U is the mid-point of $\overline{TV}$. Therefore

$$x = \frac{x_1 + x_2}{2}.$$

In the same way, projecting into the y-axis, we can show that

$$y = \frac{y_1 + y_2}{2}.$$

Thus we have proved:

Theorem 17-5. (The Mid-Point Formula.) Let $P_1 = (x_1,y_1)$ and let $P_2 = (x_2,y_2)$. Then the mid-point of $\overline{P_1P_2}$ is the point

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Problem Set 17-7

1. Visualize the points whose coordinates are listed below and compute mentally the coordinates of the mid-point of the segment joining them.
   - a. $(0,0)$ and $(0,12)$.
   - b. $(0,0)$ and $(-5,0)$.
   - c. $(1,0)$ and $(3,0)$.
   - d. $(0,-7)$ and $(0,7)$.
   - e. $(4,4)$ and $(-4,-4)$.
2. Use the mid-point formula to compute the coordinates of the mid-point of the segments joining points with the following coordinates.
   - a. $(5,7)$ and $(11,17)$.
   - b. $(-9,3)$ and $(-2,-6)$.
   - c. $(\frac{1}{2},\frac{1}{5})$ and $(\frac{1}{3},\frac{1}{8})$.

d. (2.51,-1.33) and (0.65,3.55).

e. (a,0) and (b,c).

f. (r + s,r - s) and (-r,s).

3. a. One end-point of a segment is (4,0); the mid-point is (4,1). Visualize the location of these points and state, without applying formulas, the coordinates of the other end-point.

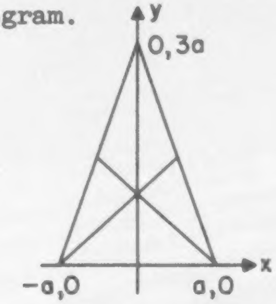
b. One end-point of a segment is (13,19). The mid-point is (-9,30). Compute the x and y coordinates of the other end-point by the appropriate formulas.

4. A quadrilateral is a square if its diagonals are congruent, perpendicular, and bisect each other. Show this to be the case for the quadrilateral having vertices, A(2,1), B(7,4), C(4,9), and D(-1,6).

5. If the vertices of a triangle are A(5,-1), B(1,5) and C(-3,1), what are the lengths of its medians?

6. Given the quadrilateral joining A(3,-2), B(-3,4), C(1,8) and D(7,4), show that the quadrilateral formed by joining its mid-points in order is a parallelogram.

7. Using coordinates, prove that two of the medians of the triangle with vertices (a,0), (-a,0) and (0,3a) are perpendicular to each other.



8. Relocate point P in the figure preceding Theorem 17-5, so that $PP_1 = \frac{1}{3}P_1P_2$ and find formulas for the coordinates of P in terms of the coordinates of P_1 and P_2. (P is between P_1 and P_2, and $x_2 > x_1$.)

*9. a. Prove: If $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P = (x,y)$ and if P is between P_1 and P_2 such that $\frac{PP_1}{PP_2} = \frac{r}{s}$, then $x = \frac{rx_2 + sx_1}{r + s}$ and $y = \frac{ry_2 + sy_1}{r + s}$.

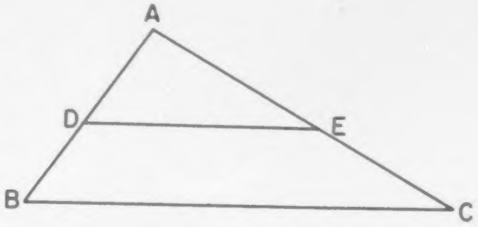
b. Use the result of part (a) to find a point P on the segment joining $P_1(5,11)$ and $P_2(25,36)$ such that $\frac{PP_1}{PP_2} = \frac{3}{5}$.

17-8. Proofs of Geometric Theorems.

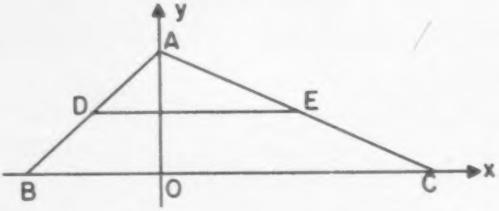
Let us now put our coordinate systems to work in proving a few geometric theorems. We start with a theorem that we have already proved by other methods.

Theorem A. The segment between the mid-points of two sides of a triangle is parallel to the third side and half as long.

Restatement: In ΔABC let D and E be the mid-points of $\overline{AB}$ and $\overline{AC}$. Then $\overline{DE} \parallel \overline{BC}$ and $DE = \frac{1}{2}BC$.



Proof: The first step in using coordinates to prove a theorem like this is to introduce a suitable coordinate system. That is, we must decide which line is to be the x-axis, which the y-axis, and which direction to take as positive along each axis. We have many choices, and sometimes a clever choice can greatly simplify our work. In the present case it seems reasonably simple to take $\overleftrightarrow{BC}$ as our x-axis, with $\overrightarrow{BC}$ as the positive direction. The y-axis we take to pass through A, with $\overrightarrow{OA}$ as the positive direction, like this:



The next step is to determine the coordinates of the various points of the figure. The x-coordinate of A is zero; the y-coordinate could be any positive number, so we write $A = (0,p)$, with the only restriction on p being $p > 0$. Similarly, $B = (q,0)$ and $C = (r,0)$, with $r > q$. (Note that we might have any of the cases $q < r < 0$, $q < r = 0$, $q < 0 < r$, $0 = q < r$, $0 < q < r$. Our figure illustrates the third case.) The coordinates of D and E can now be found by the mid-point formula. We get

$$D = (\tfrac{q}{2},\tfrac{p}{2}), \quad E = (\tfrac{r}{2},\tfrac{p}{2}).$$

Therefore the slope of $\overline{DE}$ is

$$\frac{\frac{p}{2} - \frac{p}{2}}{\frac{r}{2} - \frac{q}{2}} = \frac{0}{\frac{r - q}{2}} = 0,$$

(since $q \neq r$ the denominator is not zero).
Likewise, the slope of $\overline{BC}$ is

$$\frac{0 - 0}{\frac{r}{2} - \frac{q}{2}} = 0;$$

and so $\overline{DE} \parallel \overline{BC}$. Finally, by the distance formula,

$$DE = \sqrt{(\tfrac{r}{2} - \tfrac{q}{2})^2 + (\tfrac{p}{2} - \tfrac{p}{2})^2} = \frac{r - q}{2},$$

and

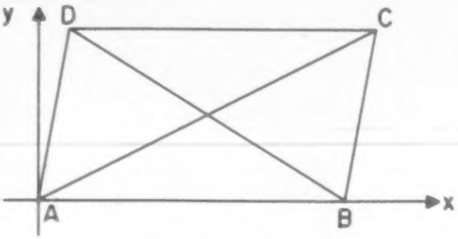
$$BC = \sqrt{(r - q)^2 + (0 - 0)^2} = r - q,$$

so that $DE = \frac{1}{2}BC$.

The algebra in this proof can be made even easier by a simple device. Instead of setting $A = (0,p)$, $B = (q,0)$, $C = (r,0)$ we could just as well have put $A = (0,2p)$, $B = (2q,0)$, $C = (2r,0)$; that is, take p, q and r as half the coordinates of the points A, B and C. If we do it this way, then no fractions arise when we divide by 2 in the mid-point formula. This sort of thing happens fairly often; foresight at the beginning can take the place of patience later on.

Theorem B. If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle.

Restatement: Let ABCD be a parallelogram, and let $AC = BD$. Then ABCD is a rectangle.



Proof: Let us take the axes as shown in the figure. Then $A = (0,0)$, and $B = (p,0)$ with $p > 0$. If we assume nothing about the figure except that ABCD is a parallelogram D could be anywhere in the upper half-plane, so that $D = (q,r)$ with $r > 0$, but no other restriction on q or r. However, C is now determined by the fact that ABCD is a parallelogram. It is fairly obvious (see the preceding proof for details) that for $\overline{DC}$ to be parallel to $\overline{AB}$ we must have $C = (s,r)$. s can be determined by the condition $\overline{BC} \parallel \overline{AD}$, like this:

$$\text{slope of } \overline{BC} = \text{slope of } \overline{AD},$$

$$\frac{r-0}{s-p} = \frac{r-0}{q-0}, \quad \text{or} \quad \frac{r}{s-p} = \frac{r}{q},$$

$$rq = r(s-p),$$

$$q = s - p, \quad (\text{since } r \neq 0)$$

$$s = p + q.$$

(The coordinates $(p + q, r)$ for C can be written down by inspection if one is willing to assume earlier theorems about parallelograms, for example, that $ABCD$ is a parallelogram if $\overline{AB} \parallel \overline{CD}$ and $AB = CD$.)

Now we finally put in the condition that $AC = BD$. Using the distance formula, we get

$$\sqrt{(p + q - 0)^2 + (r - 0)^2} = \sqrt{(q - p)^2 + (r - 0)^2}.$$

Squaring gives

$$(p + q)^2 + r^2 = (q - p)^2 + r^2,$$

$$p^2 + 2pq + q^2 + r^2 = q^2 - 2pq + p^2 + r^2,$$

or

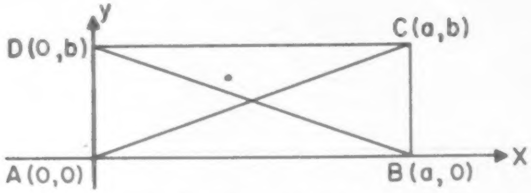
$$4pq = 0.$$

Now $4 \neq 0$ and $p \neq 0$; hence, $q = 0$. This means that D lies on the y-axis, so that $\angle BAD$ is a right angle and $ABCD$ is a rectangle.

Problem Set 17-8

Prove the following theorems using coordinate geometry:

1. The diagonals of a rectangle have equal lengths. (Hint: Place the axes as shown.)

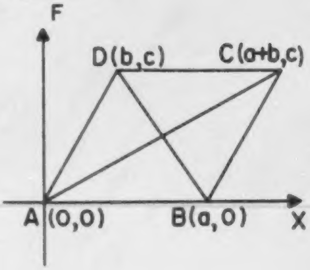


2. The mid-point of the hypotenuse of a right triangle is equidistant from its three vertices.

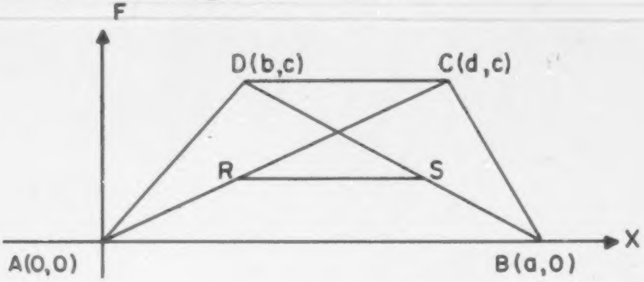
3. Every point on the perpendicular bisector of a segment is equidistant from the ends of the segment. (Hint: Select the axis in a position which will make the algebraic computation as simple as possible.)

4. Every point equidistant from the ends of a segment lies on the perpendicular bisector of the segment.

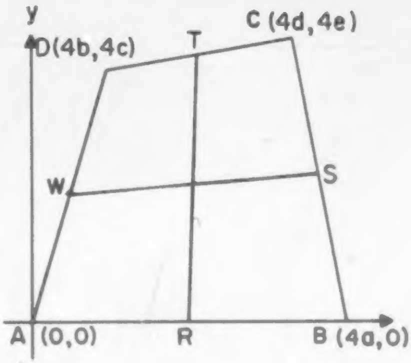
5. The diagonals of a parallelogram bisect each other. (Hint: Give the vertices of parallelogram ABCD the coordinates shown in the diagram. Show that both diagonals have the same mid-point.)

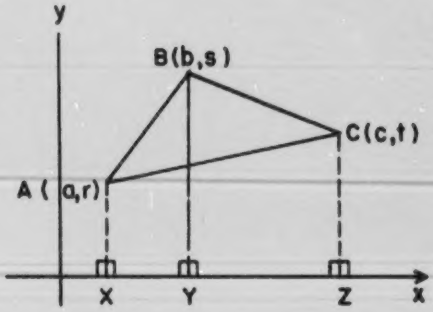


6. The line segment joining the mid-points of the diagonals of a trapezoid is parallel to the bases and equal in length to half the difference of their lengths.
In the figure R and S are mid-points of the diagonals $\overline{AC}$ and $\overline{BD}$ of trapezoid ABCD.

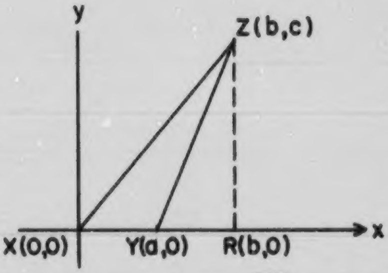


7. The segments joining mid-points of opposite sides of any quadrilateral bisect each other. (The 4's in the diagram are suggested by the fact the mid-points of segments joining mid-points must be found.)





8. The area of Δ ABC is $\frac{a(t-s)+b(r-t)+c(s-r)}{2}$, where $A = (a,r)$, $B = (b,s)$ and $C = (c,t)$. (Hint: Find three trapezoids in the figure.)



9. Given: In Δ XYZ, $\angle X$ is acute and $\overline{ZR}$ is an altitude.

   Prove: $ZY^2 = XZ^2 + XY^2 - 2XY \cdot XR$.

10. If ABCD is any quadrilateral with diagonals, $\overline{AC}$ and $\overline{BD}$, and if M and N are the mid-points of these diagonals, then $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$.

11. In Δ ABC, $\overline{CM}$ is a median to side $\overline{AB}$.

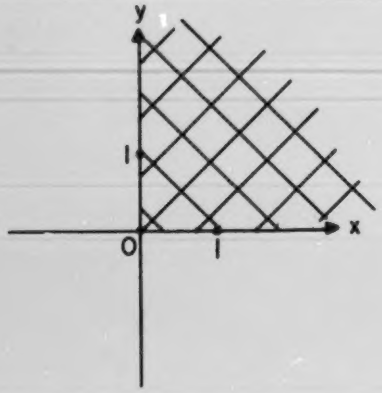
    Prove: $AC^2 + BC^2 = \frac{AB^2}{2} + 2MC^2$.

17-9. The Graph of a Condition.

By a graph we mean simply a figure in the plane, that is, a set of points. For example, triangles, rays, lines and half-planes are graphs. We can describe a graph by stating a condition which is satisfied by all points of the graph, and by no other points. Here are some examples showing a condition, a description of the graph, and the figure for each:

Condition	Graph
1. Both of the coordinates of the point P are positive.	1. The first quadrant.
2. The distance OP is 2.	2. The circle with center at the origin, and radius 2.
3. $OP < 1$.	3. The interior of the circle with center at the origin and radius 1.
4. $x = 0$.	4. The y-axis.
5. $y = 0$.	5. The x-axis.
6. $x \geq 0$ and $y = 0$.	6. The ray $\overrightarrow{OA}$, where $A = (1,0)$.
7. $x = 0$ and $y \leq 0$.	7. The ray $\overrightarrow{OB}$, where $B = (0,-1)$.

The seven graphs look like this:



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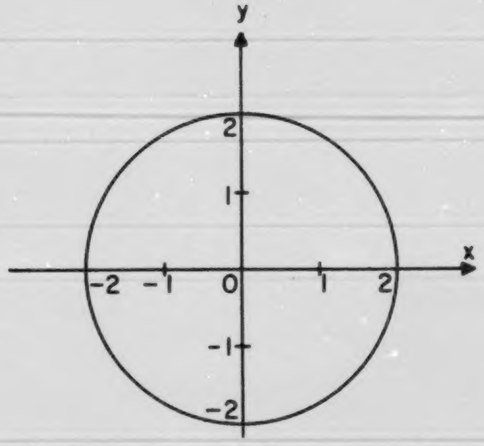

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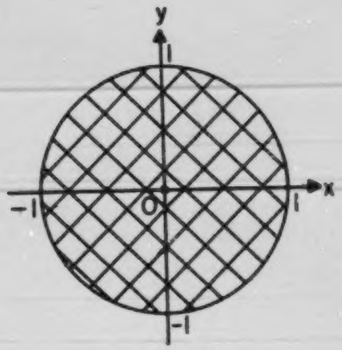

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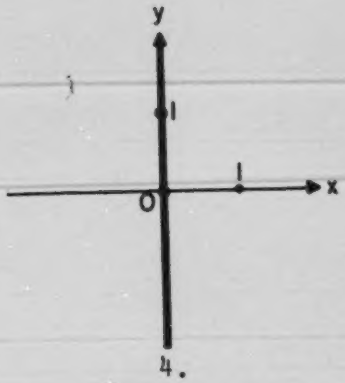

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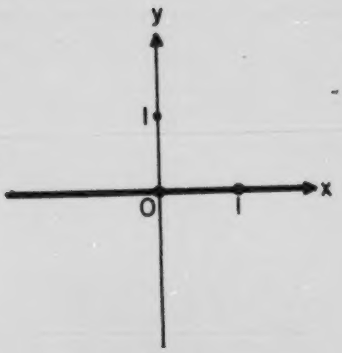

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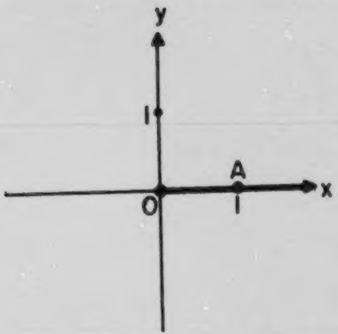

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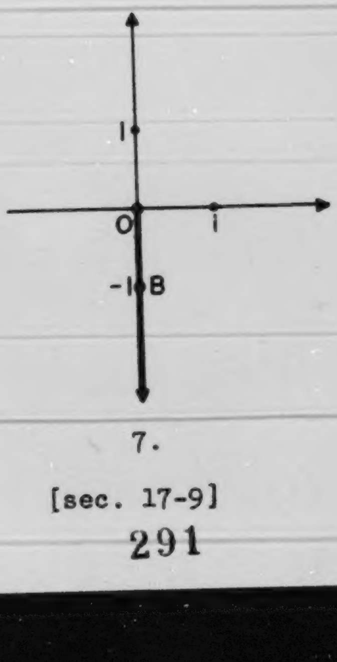


7.

You should check carefully, in each of these cases, that the graph is really accurately described by the condition in the left-hand column above. Notice that we use diagonal cross-hatching to indicate a region.

If a graph is described by a certain condition, then the graph is called the graph of that condition. For example, the first quadrant is the graph of the condition $x > 0$ and $y > 0$; the circle in Figure 2 is the graph of the condition $OP = 2$; the y-axis is the graph of the condition $x = 0$; the x-axis is the graph of the condition $y = 0$; and so on.

Very often the condition describing a graph will be stated in the form of an equation. In these cases we naturally speak of the graph of the given equation.

If you remember Chapter 14, you have probably noticed that we are doing the same thing here that we did in Sections 14-1 and 14-2, namely, characterizing a set by a property of its points. The fact that here we use the word "graph" instead of "set" is not important; it is simply customary to use the word "graph" when working with coordinate systems.

Problem Set 17-9

Sketch and describe the graphs of the conditions stated below:

1. a. $x = 5$.
   b. $|x| = 5$.
2. a. $y > 3$.
   b. $|y| < 3$.
3. $0 < x < 2$.
4. $-1 \leq x \leq 5$.
5. $-2 \leq y < 2$.
6. $x < 0$ and $y > 0$.

7. $x > 3$ and $y < -1$.

8. a. x is a positive integer.

   b. y is a positive integer.

   c. Both x and y are positive integers.

9. $x > 0$, $y > 0$, and $y > x$.

10. $1 \leq x \leq 3$ and $1 \leq y \leq 5$.

*11. $|x| < 4$ and $|y| < 4$.

*12. $|x| < 4$ and $|y| = 4$.

*13. $y = |x|$.

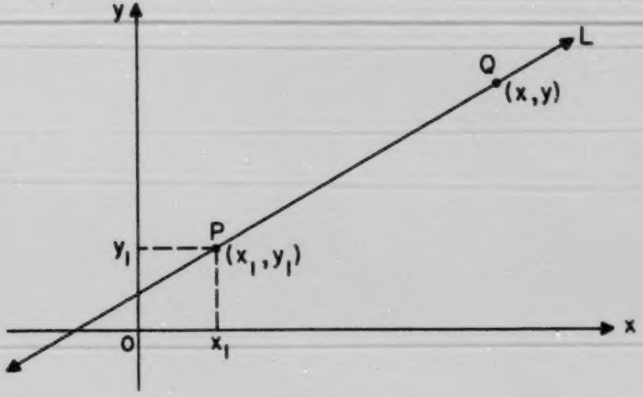
*14. $|x| = |y|$.

*15. $|x| + |y| = 5$.

17-10. How to Describe a Line by an Equation.

We are going to show that any line is the graph of a simple type of equation. We start by considering the condition which characterizes the line.

Consider a non-vertical line L, with slope m. Let P be a point of L, with coordinates (x_1, y_1).



Suppose that Q is some other point of L, with coordinates (x,y). Since $\overline{PQ}$ lies in L the slope of $\overline{PQ}$ must be m, and the coordinates of Q must satisfy the condition

$$\frac{y - y_1}{x - x_1} = m.$$

Notice that this equation is not satisfied by the coordinates of the point P, because when $x = x_1$ and $y = y_1$, the left-hand side of the equation becomes the nonsensical expression $\frac{0}{0}$, which is not equal to m (or to anything else, for that matter). If we multiply both sides of this equation by $x - x_1$ with $x \neq x_1$, we get $y - y_1 = m(x - x_1)$.

This equation is still satisfied for every point on the line different from P. And it is also satisfied for the point P itself, because when $x = x_1$ and $y = y_1$, the equation takes the form $0 = 0$, which is a true statement.

This is summarized in the following theorem:

Theorem 17-6. Let L be a non-vertical line with slope m, and let P be a point of L, with coordinates (x_1,y_1). For every point $Q = (x,y)$ of L, the equation $y - y_1 = m(x - x_1)$ is satisfied.

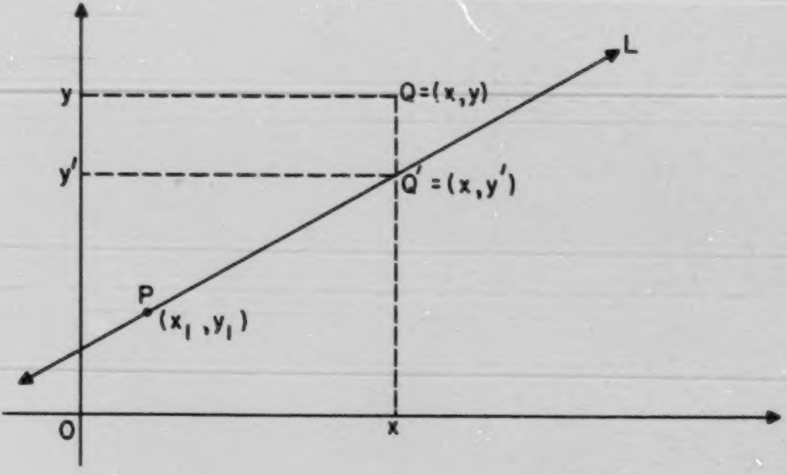
You might think at first that we have proved that the line L is the graph of the equation $y - y_1 = m(x - x_1)$. But to know that the latter is true we need to know that (compare with Section 14-1):

(1) Every point on L satisfies the equation;

(2) Every point that satisfies the equation is on L.

We have only shown (1), so we have still to show (2). We shall do this indirectly, by showing that if a point is not on L then it does not satisfy the equation.

Suppose that $Q = (x,y)$ is not on L. Then there is a point $Q' = (x,y')$ which is on L, with $y' \neq y$, like this:



By Theorem 17-1, $$\frac{y' - y_1}{x - x_1} = m;$$

hence $$y' = y_1 + m(x - x_1).$$

Since $y' \neq y$, this means that

$$y \neq y_1 + m(x - x_1).$$

Therefore $$y - y_1 \neq m(x - x_1).$$

Therefore the equation is satisfied only by points of the line.

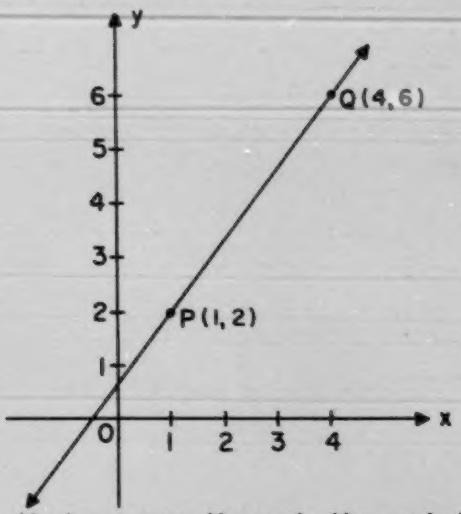
We have now proved the very important theorem:

Theorem 17-7. The graph of the equation

$$y - y_1 = m(x - x_1)$$

is the line that passes through the point (x_1,y_1) and has slope m.

The equation given in Theorem 17-7 is called the point-slope form of the equation of the line. Let us take an example:



Here we have a line that passes through the points $P = (1,2)$ and $Q = (4,6)$. The slope is

$$m = \frac{6-2}{4-1} = \frac{4}{3}.$$

Using $P = (1,2)$ as the fixed point, we get the equation

$$(1) \qquad y - 2 = \frac{4}{3}(x - 1).$$

(Here $y_1 = 2$, $x_1 = 1$, and $m = \frac{4}{3}$.) In an equivalent form, this becomes (2) $3y - 6 = 4x - 4$, (How?)

or (3) $4x - 3y = -2$.

Notice, however, that while Equation (3) is simpler to look at if all we want to do is look at it, the Equation (1) is easier to interpret geometrically. Theorem 17-7 tells us that the graph of the Equation (1) is the line that passes through the point $P = (1,2)$ and has slope $\frac{4}{3}$.

The student can readily verify that we will get the same or an equivalent equation if we had used Q as the fixed point instead of P.

Given an equation in the point-slope form, it is easy to see what the line is. For example, suppose that we have given the equation

$$y - 2 = 3(x - 4).$$

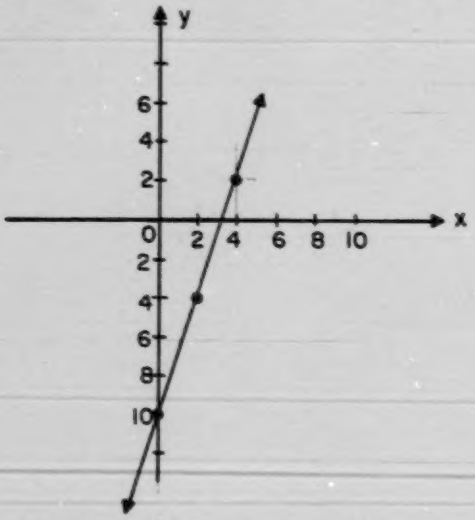
The line contains the point $(4,2)$, and has slope $m = 3$. To draw a line on graph paper, we merely need to know the coordinates of one more point. If $x = 0$, then

$$y - 2 = -12,$$

and

$$y = -10.$$

Therefore, the point $(0,-10)$ is on the line, and we can complete the graph:

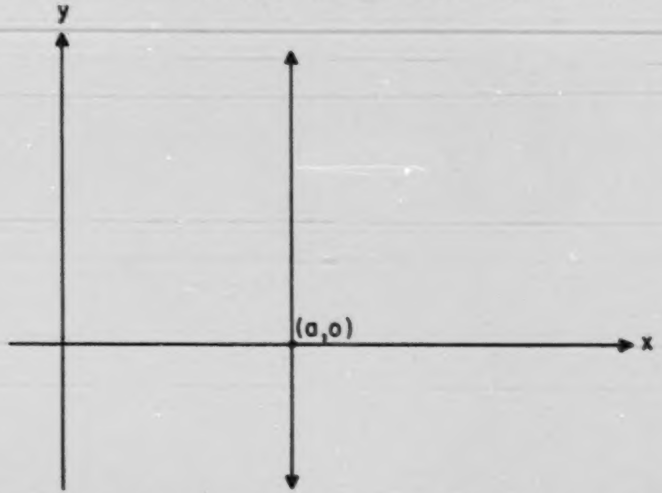


Logically speaking, this is all that we need. As a practical matter, it is a very good idea to check the coordinates of one more point. This point can be selected anywhere along the line, but to serve as a good check it should not be too near the other two points. If we take $x = 2$, we get

$$y - 2 = -6, \quad \text{or} \quad y = -4.$$

As well as we can judge from the figure, the point $(2,-4)$ lies on the line.

At the beginning of this section we promised to show that any line is the graph of a simple type of equation. We have shown this for any non-vertical line, but we must still consider a vertical line. Suppose a vertical line crosses the x-axis at the point with coordinates $(a,0)$, as in the figure.



Since the vertical line is perpendicular to the x-axis, every point of the line has its x-coordinate equal to a. Furthermore, any point not on the line will have its x-coordinate not equal to a. Hence, the condition which characterizes the vertical line is $x = a$, certainly a very simple type of equation.

Problem Set 17-10

In each of the following problems, we have given the coordinates of a point P and the value of the slope m. Write the point-slope form of the equation of the corresponding line, and draw the graph. Check your work by checking the coordinates of at least one point that was not used in plotting the line. It is all right to draw several of these graphs on the same set of axes, as long as the figures do not become too crowded.

1. $P = (-1,2)$, $m = 4$.
2. $P = (1,-1)$, $m = -1$.
3. $P = (0,5)$, $m = -\frac{1}{3}$.
4. $P = (-1,-4)$, $m = \frac{5}{2}$.
5. $P = (3,-2)$, $m = 0$.

By changing to a point-slope form where necessary, show that the graph of each of the following equations is a line. Then draw the graph and check, as in the preceding problems.

6. $y - 1 = 2(x - 4)$.
7. $y = 2x - 7$.
8. $2x - y - 7 = 0$.
9. $y + 5 = \frac{1}{3}(x + 3)$.
10. $x - 3y = 12$.
11. $y = x$.
12. $y = 2x$.
13. $y = 2x - 6$.
14. $y = 2x + 5$.
15. $x = 4$.
16. $x = 0$.
17. $y = 0$.

18. Thinking in three-dimensional coordinates, describe in words the set of points represented by the following equations. For example, $y = 0$ is the equation of the xz-plane, that is, the plane determined by the x and z- axes. (Refer to Problem 12 of Problem Set 17-3.)

a. $x = 0$. c. $x = 1$.

b. $z = 0$. d. $y = 2$.

17-11. Various Forms of the Equation of a Line.

We already know how to write an equation for a non-vertical line if we know the slope m and the coordinates (x_1,y_1) of one point of the line. In this case we know that the line is the graph of the equation

$$y - y_1 = m(x - x_1),$$

in the point-slope form.

Definition: The point where the line crosses the y-axis is called the y-intercept. If this is the point $(0,b)$, then the point-slope equation takes the form

$$y - b = m(x - 0),$$
$$y = mx + b.$$

This is called the slope-intercept form. The number b is also called the y-intercept of the line. (When we see the phrase y-intercept, we will have to tell from the context whether the number b or the point $(0,b)$ is meant.) Thus we have the following theorem:

Theorem 17-8. The graph of the equation

$$y = mx + b$$

is the line with slope m and y-intercept b.

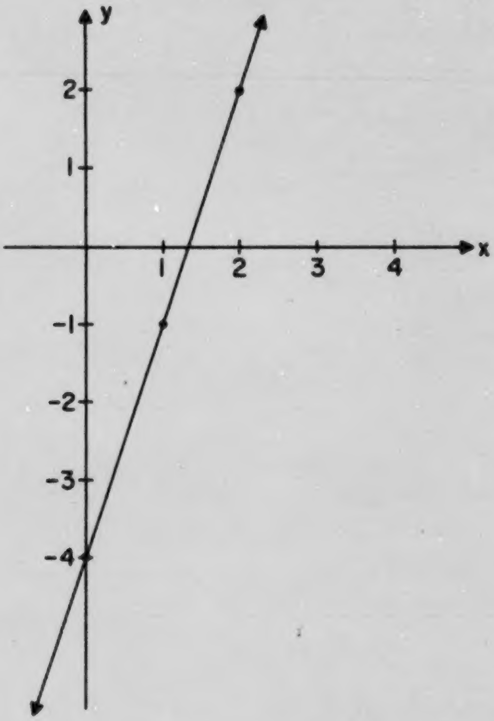
If we have an equation given in this form, then it is easy to draw the graph. All we need to do is to give x any value other than 0, and find the corresponding value of y. We then have the coordinates of two points on the line, and can draw the line. For example, suppose that we have given

$$y = 3x - 4.$$

Obviously the point $(0,-4)$ is on the graph. Setting $x = 2$, we get

$$y = 6 - 4 = 2.$$

Therefore the point $(2,2)$ is on the line, and the line therefore looks like this:



As a check, we find that for $x = 1$,

$$y = 3 - 4 = -1,$$

and the point $(1,-1)$ lies on the graph, as well as we can judge.

Notice that once we have Theorem 17-8, we can prove that certain equations represent lines, by converting them to the slope-intercept form. For example, suppose we have given

$$(1) \qquad 3x + 2y + 4 = 0.$$

This is algebraically equivalent to the equation

$$2y = -3x - 4,$$

or

$$(2) \qquad y = -\frac{3}{2}x - 2.$$

Being equivalent, Equations (1) and (2) have the same graph. The graph of (2) is a line, namely, the line with slope $m = -\frac{3}{2}$ and y-intercept $b = -2$. The graph of (1) is the same line.

17-12. The General Form of the Equation of a Line.

Theorem 17-8, of course, applies only to non-vertical lines, because these are the ones that have slopes. Vertical lines are very simple objects, algebraically speaking, because they are the graphs of simple equations, of the form

$$x = a.$$

Thus we have two kinds of equations ($y = mx + b$ and $x = a$) for non-vertical and vertical lines respectively. We can tie all this together, including both cases, in the following way.

Definition: By a linear equation in x and y we mean an equation of the form

$$Ax + By + C = 0,$$

where A and B are not both zero.

The following two theorems describe the relation between geometry and algebra, as far as lines are concerned:

Theorem 17-9. Every line in the plane is the graph of a linear equation in x and y.

Theorem 17-10. The graph of a linear equation in x and y is always a line.

Now that we have got this far, both of these theorems are very easy to prove.

Proof of Theorem 17-9: Let L be a line in the plane. If L is vertical, then L is the graph of an equation

$$x = a,$$

or

$$x - a = 0.$$

This has the form $Ax + By + C = 0$, where $A = 1$, $B = 0$, $C = -a$. A and B are not both zero, because $A = 1$, and so the equation is linear.

If L is not vertical, then L has a slope m and crosses the y-axis at some point $(0,b)$. Therefore L is the graph of the equation

$$y = mx + b,$$

or

$$mx - y + b = 0.$$

This has the form $Ax + By + C = 0$, where $A = m$, $B = -1$, $C = b$. A and B are not both zero, because $B = -1$. Therefore the equation is linear. (Notice that it can easily happen that $m = 0$; this holds true for all horizontal lines. Notice also that the equation is not unique: e.g. $2Ax + 2By + 2C = 0$ has the same graph as $Ax + By + C = 0$.)

Proof of Theorem 17-10: Given the equation $Ax + By + C = 0$ with A and B not both zero.

Case 1. If $B = 0$, then the equation has the form

$$Ax = -C.$$

Since $B = 0$, we know that $A \neq 0$. Therefore we can divide by A, getting

$$x = \frac{C}{A}.$$

The graph of this equation is a vertical line.

Case 2. Suppose that $B \neq 0$. Then we can divide by B, getting

$$\frac{A}{B}x + y + \frac{C}{B} = 0,$$

or

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

The graph of this equation is a line, namely, the line with slope $m = -\frac{A}{B}$ and y-intercept $b = -\frac{C}{B}$.

To make sure that you understand what has been proved, in Theorems 17-9 and 17-10, you should notice carefully a certain thing that has not been proved. We have not proved that if a given equation has a line as its graph, then the equation is linear. And in fact this latter statement is not true. For example, consider the equation

$$x^2 = 0.$$

Now the only number whose square is zero is the number zero itself. Therefore the equation $x^2 = 0$ says the same thing as the equation $x = 0$. Therefore the graph of the equation $x^2 = 0$ is the y-axis, which is of course a line. Similarly, the graph of the equation

$$y^{17} = 0$$

is the x-axis.

The same sort of thing can happen in cases where it is not so easy to see what is going on. For example, take the equation

$$x^2 + y^2 = 2xy.$$

This can be written in the form

$$x^2 - 2xy + y^2 = 0,$$

or

$$(x - y)^2 = 0.$$

The graph is the same as the graph of the equation

$$x - y = 0,$$

or

$$y = x.$$

The graph is a line.

Notice that the proof of Theorem 17-10 gives us a practical procedure for getting information about the line from the general equation. If $B = 0$, then we have the vertical line given by the equation

$$x = -\frac{C}{A}.$$

Otherwise, we solve for y, getting

$$y = -\frac{A}{B}x - \frac{C}{B},$$

where the slope is

$$m = -\frac{A}{B}$$

and the y-intercept is

$$b = -\frac{C}{B}.$$

Problem Set 17-12

Sketch the graphs of the following equations:

1. $2x + 5y = 7.$
2. $\frac{1}{2}y - 2x + 3 = 0.$
3. $x + 4 = 0.$
4. $y + 4 = 0.$

Describe the graphs of the following equations:

5. $0 \cdot x + 0 \cdot y = 0.$
6. $0 \cdot x + 0 \cdot y = 2.$
7. $x^2 + y^2 = 0.$
8. $x^2 = -1.$

Sketch the graphs of the following conditions:

9. $3x + 4y = 0$ and $x \leq 0.$
10. $5x - 2y = 0$ and $5 \leq y \leq 10.$
11. $(x + y)^2 = 0.$
12. $(y - 1)^{54} = 0.$

Find linear equations $(Ax + By + C = 0)$ of which the following lines are the graphs. State the values for A, B, C in your answer.

13. The line through (1,2) with slope 3.
14. The line through (1,0) and (0,1).
15. The line with slope 2 and y-intercept -4.
16. The x-axis.
17. The y-axis.
18. The horizontal line through (-5,-3).
19. The vertical line through (-5,-3).
20. The line through the origin and the mid-point of the segment with end-points (3,2) and (7,0).

17-13. Intersections of Lines.

Suppose that we have given the equations of two lines, like this:

$$L_1: 2x + y = 4,$$

$$L_2: x - y = -1.$$

These lines are not parallel, because the slope of the first is $m_1 = -2$, and the slope of the second is $m_2 = 1$. Therefore, they intersect in some point $P = (x,y)$. The pair of numbers (x,y) must satisfy both equations. Therefore the geometric problem of finding the point P is equivalent to the algebraic problem of solving a system of two linear equations in two unknowns.

To solve the system is easy. Adding the two equations, we get

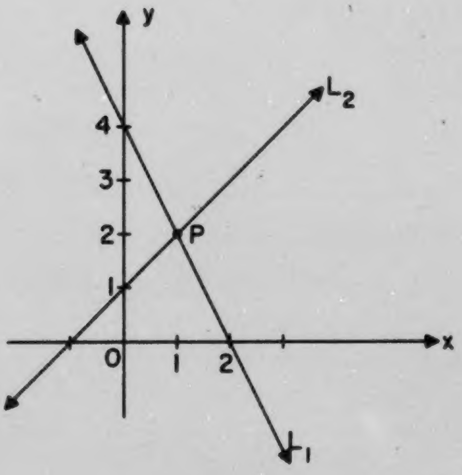
$$3x = 3,$$

or

$$x = 1.$$

Substituting 1 for x in the second equation, we get $y = 2$. The values $x = 1$, $y = 2$ will also satisfy the first equation. Do they?

Therefore $P = (1,2)$. The graph makes this look plausible:



This method always gives the answer to our problem, whenever our problem has an answer, that is, whenever the graphs of the two equations intersect. If the lines are parallel, then the corresponding system of equations will be inconsistent, that is, the solution of the system will be the empty set. This will be plain enough when we try to solve the system.

Problem Set 17-13

1. Find the common solution of the following pairs of equations and draw their graphs.

   a. $y = 2x$ and $x + y = 7$.

   b. $y = 2x$ and $y - 2x = 3$.

   c. $x + y = 3$ and $2y = 6 - 2x$.

2. a. The graphs of which pairs of the equations listed below would be parallel lines?

   b. Intersecting but not coincident lines?

   c. Coincident lines?

   The equations are

   (1) $y = 3x + 1$.

   (2) $y = 4x + 1$.

   (3) $2y = 6x + 2$.

   (4) $y - 3x = 2$.

3. Suppose the unit in our coordinate system is 1 mile. How many miles from the origin is the point where the line $y = \frac{1}{1000}x - 4$ crosses the x-axis?

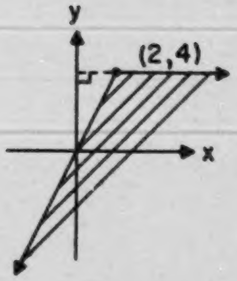
4. Find the intersection of the graphs of the following pairs of conditions:

   a. $y = 2x$ and $y = 4$.

   b. $y = 2x$ and $y \geq 4$.

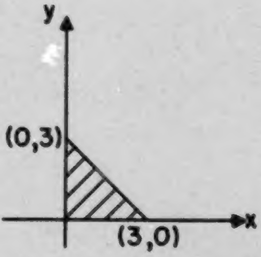
   c. $y < 2x$ and $y > 4$.

   d. What pair of conditions will determine the interior of the angle shown in the figure?



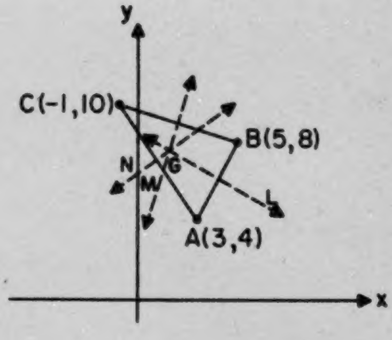
5. a. Sketch the intersection of the graphs of all three conditions $x + y > 3$, $y < 4$, $x < 2$.

   b. State the three conditions which would determine the interior of the triangle shown.



6. Find an equation for the perpendicular bisector of the segment with end-points $(3,4)$ and $(5,8)$.

7. Find equations for the perpendicular bisectors of the sides of $\Delta\ (3,4)(5,8)(-1,10)$, and show that they intersect in a point.

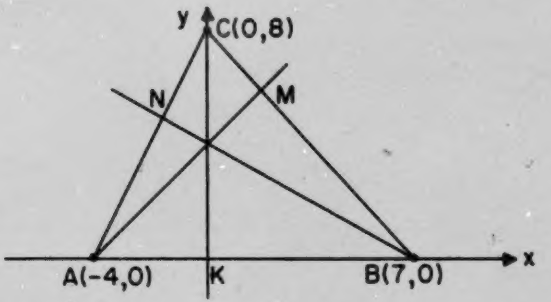


*8. The following instructions were found on an ancient document. "Start from the crossing of King's Road and Queen's Road. Proceeding north on King's Road, find first a pine tree, then a maple. Return to the crossing. West on Queen's Road there is an elm and east on Queen's Road there is a spruce. One magical point is where the elm-pine line meets the maple-spruce line. The other magical point is where the spruce-pine line meets the elm-maple line. The line joining the two magical points meets Queen's Road where the treasure is buried."

A search party found the elm 4 miles from the crossing, the spruce 2 miles from the crossing, and the pine 3 miles from the crossing, but there was no trace of the maple. Nevertheless they were able to find the treasure from the instructions. Show how this was done.

One man in the party remarked on how fortunate they were to have found the pine still standing. The leader laughed and said, "We didn't need the pine tree either." Show that he was right.

*9. One of the altitudes of the $\triangle ABC$, where $A = (-4,0)$, $B = (7,0)$, $C = (0,8)$, is the y-axis. Why? Prove, using coordinate methods, that the altitudes from A and B meet on that axis. (Hint: Find the intersections of those altitudes with the y-axis.)
Do the same for the triangle with vertices $(a,0)$, $(b,0)$, $(0,c)$.



*10. The centroid of a triangle is defined as the intersection of the three medians. Prove that the coordinates of the centroid are just the averages of the coordinates of the vertices.

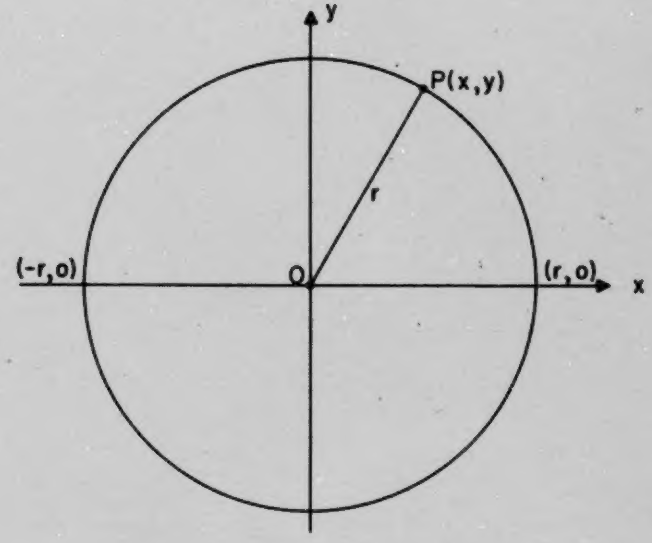
*11. Find the distance from the point $(1,2)$ to the line $x + 3y + 1 = 0$.

*12. Find the distance from the point (a,b) to the line $y = x$.

*13. In the general case of the triangle of Problem 9, let H be the point of concurrence of the altitudes, M the point of concurrence of the medians, and D the point of concurrence of the perpendicular bisectors of the sides. Prove, using Problems 9 and 10 that these three points are collinear, and that M divides $\overline{DH}$ in the ratio two to one (refer to Problem 8 of Problem Set 17-7).

17-14. Circles.

Consider the circle with center at the origin and radius r.



This figure is defined by the condition

$$OP = r.$$

Algebraically, in terms of the distance formula, this says that

$$\sqrt{(x - 0)^2 + (y - 0)^2} = r,$$

or

$$x^2 + y^2 = r^2.$$

That is, if $P(x,y)$ is a point of the circle then $x^2 + y^2 = r^2$. We still have to show that if $x^2 + y^2 = r^2$ then $P(x,y)$ is a point of the circle. This we do by reversing the algebraic steps:

If

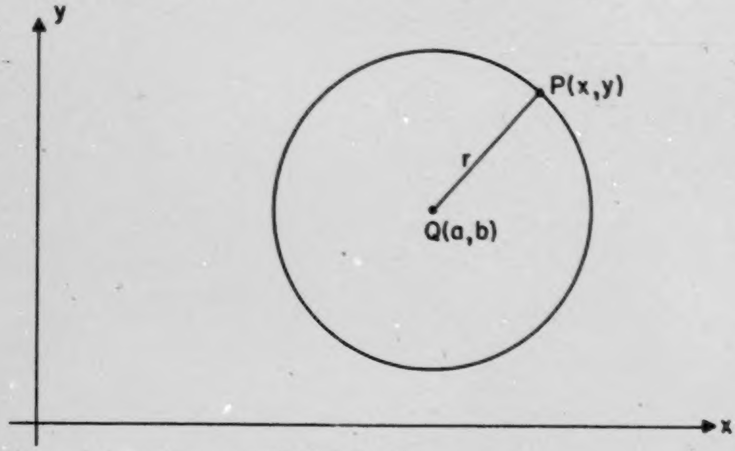
$$x^2 + y^2 = r^2$$

then

$$\sqrt{(x - 0)^2 + (y - 0)^2} = r,$$

since r is a positive number. This equation says that $OP = r$, and so P is a point of the circle.

Consider, more generally, the circle with center at the point $Q = (a,b)$ and radius r.



This is defined by the condition $QP = r$,

or

$$\sqrt{(x - a)^2 + (y - b)^2} = r,$$

or

$$(x - a)^2 + (y - b)^2 = r^2.$$

In this case, also, the algebraic steps can be reversed, and so we can say that

$$(x - a)^2 + (y - b)^2 = r^2$$

is the equation of the circle.

This is the standard form of the equation of the circle, with center (a,b) and radius r. For future reference, let us state this result as a theorem.

Theorem 17-11. The graph of the equation

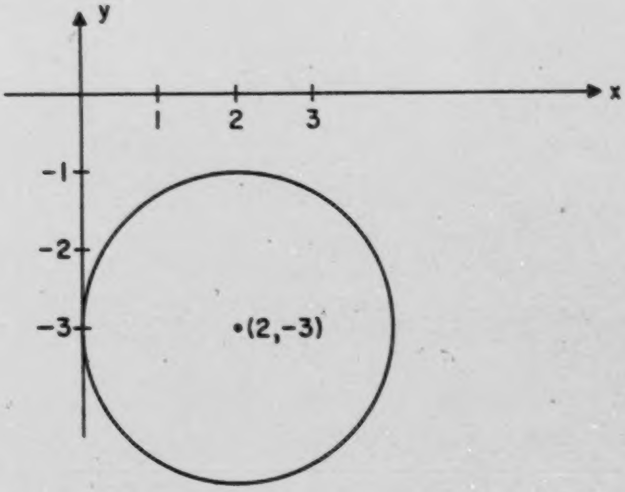
$$(x - a)^2 + (y - b)^2 = r^2$$

is the circle with center at (a,b) and radius r.

If an equation is given in this form, we can read off immediately the radius and the coordinates of the center. For example, suppose that we have given the equation

$$(x - 2)^2 + (y + 3)^2 = 4.$$

The center is the point $(2,-3)$, the radius is 2, and the circle looks like this:



So far, this is easy enough. But suppose that the standard form of the equation has fallen into the hands of someone who likes to "simplify" formulas algebraically. He would have "simplified" the equation like this:

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 4$$
$$x^2 + y^2 - 4x + 6y + 9 = 0.$$

From his final form, it is not at all easy to see what the graph is. Sometimes we will find equations given in forms like this. Therefore we need to know how to "unsimplify" these forms so as to get back the standard form

$$(x - a)^2 + (y - b)^2 = r^2.$$

The procedure is this. First we group the terms in x together, and the terms in y together, and write the equation with the constant term on the right, like this:

$$x^2 - 4x \qquad + y^2 + 6y \qquad = -9.$$

Then we see what constant should be added to the first two terms to complete a perfect square. Recall that to find this constant take half of the coefficient of x, and square the result. Here we get 4. The same process, applied to the third and fourth terms, shows that we should add 9 in order to make a perfect square. Thus we are going to add a total of 13 to the left-hand side of the equation. Therefore we must add 13 to the right-hand side. Now our equation takes the equivalent form

$$x^2 - 4x + 4 + y^2 + 6y + 9 = -9 + 13,$$

or

$$(x - 2)^2 + (y + 3)^2 = 4,$$

as before.

If we multiply out and simplify in the standard form, we get

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0.$$

This has the form

$$x^2 + y^2 + Ax + By + C = 0.$$

Thus we have the theorem:

Theorem 17-12. Every circle is the graph of an equation of the form

$$x^2 + y^2 + Ax + By + C = 0.$$

It might seem reasonable to suppose that the converse is also true. That is, we might think that every equation of the form that we have been discussing has a circle as its graph. But this is not true by any means. For example, consider the equation

$$x^2 + y^2 = 0.$$

Here A, B and C are all zero. If x and y satisfy this equation, then x and y are both zero. That is, the graph of the equation is a single point, namely, the origin.

Consider next the equation

$$x^2 + y^2 + 1 = 0.$$

Here $A = B = 0$ and $C = 1$. This equation is not satisfied by the coordinates of any point whatsoever. (Since $x^2 \geq 0$ and $y^2 \geq 0$ and $1 > 0$, it follows that $x + y + 1 > 0$ for every pair of real numbers x and y.) For this equation, the graph has no points at all, and is therefore the empty set.

In fact, the only possibilities are the circle that we would normally expect, plus the two unexpected possibilities that we have just noted.

Theorem 17-13. Given the equation

$$x^2 + y^2 + Ax + By + C = 0.$$

The graph of this equation is (1) a circle, (2) a point or (3) the empty set.

Proof: Let us complete the square for the terms in x, and complete the square for the terms in y, just as we did in the particular case that we worked out above. This gives

$$x^2 + Ax + \frac{A^2}{4} + y^2 + By + \frac{B^2}{4} = -C + \frac{A^2}{4} + \frac{B^2}{4},$$

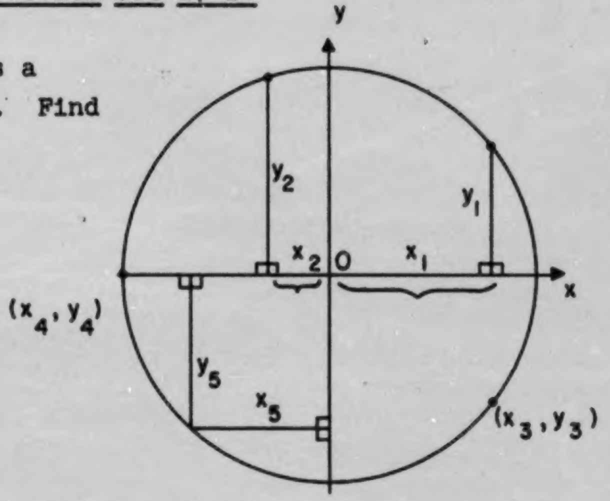
or

$$(x + \frac{A}{2})^2 + (y + \frac{B}{2})^2 = \frac{A^2 + B^2 - 4C}{4}.$$

If the fraction on the right is positive, equal to r^2 with $r > 0$, then the graph is a circle with center at $(-\frac{A}{2},-\frac{B}{2})$ and radius r. If the fraction on the right is zero, then the graph is the single point $(-\frac{A}{2},-\frac{B}{2})$. If the fraction on the right is negative, then the equation is never satisfied, and the graph contains no points at all.

Problem Set 17-14

1. The circle shown has a radius of 5 units. Find the value of:



a. $x_1^2 + y_1^2$.

b. $x_2^2 + y_2^2$.

c. $x_3^2 + y_3^2$.

d. $x_4^2 + y_4^2$.

e. $x_5^2 + y_5^2$.

2. a. Which of the following eight equations have graphs which are circles?

b. Which of the circles would have centers at the origin?

c. Which would have centers on an axis, but not at the origin?

(1) $x^2 + (y - 1)^2 = 9$.
(2) $y = x^2$.
(3) $x^2 + y^2 = 7$.
(4) $1 - x^2 = y^2$.
(5) $(x - 2)^2 - (y - 9)^2 = 16$.
(6) $(x - 2)^2 + (y - 3)^2 = 16$.
(7) $3x^2 + y^2 = 4$.
(8) $x^2 + y^2 = 0$.

3. Determine the center and radius of each of the following circles.

a. $x^2 + y^2 = 3^2$.
b. $x^2 + y^2 = 100$.
c. $(x - 1)^2 + y^2 = 16$.
d. $x^2 + y^2 = 7$.
e. $y^2 = 4 - x^2$.
f. $(x - 4)^2 + (y - 3)^2 = 36$.
g. $(x + 1)^2 + (y + 5)^2 = 49$.
h. $x^2 - 2x + 1 + y^2 = 25$.
i. $x^2 - 2x + y^2 = 24$.
j. $x^2 + 6x + y^2 - 4y = 12$.

4. A circle has the equation: $x^2 - 10x + y^2 = 0$.

   a. Show algebraically that the points $(0,0)$, $(1,3)$ and $(2,4)$ all lie on the circle.

   b. Find the center and radius of the circle.

   c. Show that if $(1,3)$ is joined to the ends of the diameter on the x-axis, a right angle is formed with vertex at $(1,3)$.

5. a. Find the points where the circle $(x - 3)^2 + y^2 = 25$ is intersected by the x- and y-axes.

   b. Considering portions of the x- and y-axes as chords of the circle in part a., prove (as you should of course, expect from Theorem 13-14) that the products of the lengths of the parts into which each chord is divided by the other, are equal.

6. Draw the four circles obtained by choosing the various possible sign combinations in

$$(x \pm 1)^2 + (y \pm 1)^2 = 1.$$

Then write the equations of the circle tangent to all four and containing them. Is there another circle tangent to all four. What is its radius?

7. Draw the 4 circles given by

$$x^2 + y^2 = \pm 10x, \qquad x^2 + y^2 = \pm 10y$$

and write the equation of a circle tangent to all of them.

8. Given the circle $x^2 + y^2 = 16$ and the point $K(-7,0)$.

   a. Find the equation (in point-slope form) of the line L_m with slope m passing through the point K.

   b. Find the points (or point) of intersection of L_m and the circle.

   c. For what values of m is there exactly one point of intersection? Interpret this result geometrically.

9. Find an equation for a circle tangent externally to the circle

$$x^2 + y^2 - 10x - 6y + 30 = 0$$

and also tangent to the x- and y-axes.

REVIEW PROBLEMS

1. What are the coordinates of the projection into the x-axis of the point $(5,2)$?

2. Three of the vertices of a rectangle are $(-1,-1)$, $(3,-1)$ and $(3,5)$. What is the fourth vertex?

3. An isosceles triangle has vertices $(0,0)$, $(4a,0)$ and $(2a,2b)$. What is the slope of the median from the origin? of the median from $(2a,2b)$?

4. In Problem 3 what is the slope of the altitude which contains the origin?

5. What is the length of each of the medians of the triangle in Problem 3?

6. What is the slope of a line that is parallel to a line which passes through the origin and through $(-2,3)$.

7. The vertices of a quadrilateral are $(0,0)$, $(5,5)$, $(7,1)$ and $(1,7)$. What are the lengths of its diagonals?

8. What are the coordinates of the mid-points of segments joining the pairs of points in Problem 7?

9. The vertices of a square are labeled consecutively, P, Q, R and S. T is the mid-point of $\overline{QR}$ and U is the mid-point of $\overline{RS}$. $\overline{PT}$ intersects $\overline{QU}$ at V.

   a. Prove that $\overline{PT} \cong \overline{QU}$.

   b. Prove that $\overline{PT} \perp \overline{QU}$.

   *c. Prove that $VS = PQ$.

   (Hint: Let $P = (0,0)$ and $Q = (2a,0)$.)

10. Use coordinate geometry to prove the theorem: The median of a trapezoid bisects a diagonal.

11. What is the equation whose graph is the y-axis?

12. A rhombus ABCD has A at the origin and $\overline{AB}$ in the positive x-axis. $m\angle A = 45$. $AB = 6$. C is in the first quadrant. What is the equation of $\overleftrightarrow{AB}$? $\overleftrightarrow{BC}$? $\overleftrightarrow{CD}$?

13. The coordinates of the vertices of a trapezoid are, consecutively, $(0,0)$, $(a,0)$, (b,c) and (d,c). Find the area of the trapezoid in terms of these coordinates.

14. The graphs of the equations $y = \frac{1}{2}x$ and $y = -2x + 5$ are perpendicular to each other at what point?

15. Name the set of points such that the sum of the squares of the distances of each point from the two axes is 4.

16. Write the equation of the circle which has

    a. its radius 7 and center at the origin.

    b. its radius k and center at the origin.

    c. its radius 3 and center at $(1,2)$.

*17. Prove that the line $x + y = 2$ is tangent to the circle $x^2 + y^2 = 2$.

Chapters 13 to 17

REVIEW EXERCISES

Write (1) if the statement is true and (0) if it is false. Be able to explain why you mark a statement false.

1. If a line through the center of a circle is perpendicular to a chord of that circle, it bisects the chord.
2. If $\overline{AB}$ is a radius of a circle and $\overleftrightarrow{CB}$ is tangent to the circle, then $\overline{AB} \perp \overleftrightarrow{CB}$.
3. A line which bisects two chords of a circle is perpendicular to both of them.
4. The intersection of the interiors of two circles may be the interior of a circle.
5. Every point in the interior of a circle is the mid-point of exactly one chord of the circle.
6. The longer an arc is, the longer its chord is.
7. If a line intersects a circle, the intersection consists of two points.
8. If a plane and a sphere intersect, and if the intersection is not a circle, it is a point.
9. If a plane is tangent to a sphere, a line perpendicular to the plane at the point of tangency contains the center of the sphere.
10. On a given circle, $m\widehat{XY} + m\widehat{YZ} = m\widehat{XZ}$.
11. A 90° inscribed angle will always intercept a 45° arc.
12. Two angles which intercept the same arc are congruent.
13. Congruent chords drawn in each of two concentric circles have congruent arcs.

14. If a triangle inscribed in a circle has no side intersecting a given diameter then the triangle contains an obtuse angle.

15. If two chords in a circle intersect, the ratio of the segments of one chord is equal to the ratio of the segments of the other chord.

16. If $\overline{AB}$ is tangent to a circle at B and if $\overleftrightarrow{AC}$ intersects the circle at C and D, then $AB^2 = AC \cdot AD$.

17. In a plane, the set of points equidistant from the ends of a segment is the perpendicular bisector of the segment.

18. The set of points one inch from a given line is a line parallel to the given line.

19. Any point in the interior of an angle which is not equidistant from the sides of the angle does not lie on the bisector of the angle.

20. The three altitudes of any right triangle are concurrent.

21. Two circles intersect if the distance between their centers is less than the sum of their radii.

22. The three angle bisectors of a triangle are concurrent at a point equidistant from the vertices of the triangle.

23. The perpendicular bisectors of two sides of a triangle may intersect outside the triangle.

24. Using straight-edge and compass, it is possible to trisect a segment.

25. In bisecting a given angle by the method shown in the text, it is necessary to draw at least four arcs.

26. The ratio of radius to circumference is the same for all circles.

27. The area of a circle of diameter d is $\frac{1}{2}\pi d^2$.

28. A plane section of a triangular prism may be a parallelogram.

29. A plane section of a triangular pyramid may be a parallelogram.

30. The volume of a triangular prism is half the product of the area of its base and its altitude.

31. In any pyramid a section made by a plane which bisects the altitude and is parallel to the base has half the area of the base.

32. Two pyramids with the same volume and the same base area have congruent altitudes.

33. The volume of a pyramid with a square base is equal to one-third of its altitude multiplied by the square of a base edge.

34. The area of the base of a cone can be found by dividing three times the volume by the altitude.

35. The radius of the base of a circular cylinder is given the formula $\sqrt{\frac{V}{\pi h}}$, where V is the volume of the cylinder and h its altitude.

36. The volume of a sphere is given by the formula $\frac{1}{6}\pi d^3$ where d is its diameter.

37. The slope of a segment depends on the quadrant or quadrants in which the segment lies.

38. If two segments have the same slope they are parallel.

39. If the slopes of two lines are -2 and .5 the lines are perpendicular.

40. If the coordinates of two points are (a,b) and (c,d), the distance between them is (d - b) + (c - a).

41. If a segment joins (r,s) to (-r,-s), then its mid-point is the origin.

42. The point (-2,-1) lies on the graph of $xy - 2x - y + 2 = 0$.

43. The distance between (3,0) and (4,0) is 5.

44. If two vertices of a right triangle have coordinates (0,10) and (8,0) the third vertex is at the origin.

45. If three vertices of a rectangle have coordinates $(0,m)$, $(r,0)$ and (r,m) the fourth vertex is at the origin.

46. The equation of a line with slope 2 and containing $(3,4)$ is $4y + 3x = 2$.

47. The x-intercept of the graph of $y = 3x + 9$ is -3.

48. The intersection of the graphs of $y = 3x + 2$ and $y = 3x + 1$ is a single point.

49. The graph of $x^2 + y^2 - 4 = 0$ is a circle.

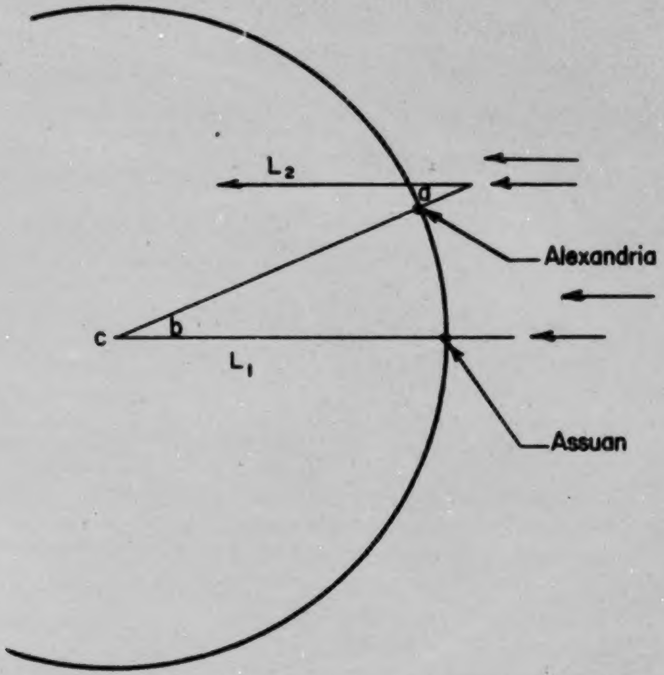
50. The graph of every condition is either a line or a curve.

Appendix VII

HOW ERATOSTHENES MEASURED THE EARTH

The circumference of the earth, at the equator, is about 40,000 kilometers, or about 24,900 miles. Christopher Columbus appears to have thought that the earth was much smaller than this. At any rate, the West Indies got their name, because when Columbus reached them, he thought that he was already in India. His margin of error, therefore, was somewhat greater than the width of the Pacific Ocean.

In the third century B.C., however, the circumference of the earth was measured, by a Greek mathematician, with an error of only one or two per cent. The man was Eratosthenes, and his method was as follows:



It was observed that at Assuan on the Nile, at noon on the Summer Solstice, the sun was exactly overhead. That is, at noon of this particular day, a vertical pole cast no shadow at all, and the bottom of a deep well was completely lit up.

In the figure, C is the center of the earth. At noon on the Summer Solstice, in Alexandria, Eratosthenes measured the angle marked a on the figure, that is, the angle between a vertical pole and the line of its shadow. He found that this angle was about $7^{\circ}12'$, or about $\frac{1}{50}$ of a complete circumference.

Now the sun's rays, observed on earth, are very close to being parallel. Assuming that they are actually parallel, it follows when the lines L_1 and L_2 in the figure are cut by a transversal, alternate interior angles are congruent. Therefore, $\angle a \cong \angle b$. Therefore, the distance from Assuan to Alexandria must be about $\frac{1}{50}$ of the circumference of the earth.

The distance from Assuan to Alexandria was known to be about 5,000 Greek stadia. (A stadium was an ancient unit of distance.) Eratosthenes concluded that the circumference of the earth must be about 250,000 stadia. Converting to miles, according to what ancient sources tell us about what Eratosthenes meant by a stadium, we get 24,662 miles.

Thus Eratosthenes' error was well under two percent. Later, he changed his estimate to an even closer one, 252,000 stadia, but nobody seems to know on what basis he made the change. On the basis of the evidence, some historians believe that he was not only very clever and very careful, but also very lucky.

Appendix VIII

RIGID MOTION

VIII-1. The General Idea of a Rigid Motion.

In Chapters 5 and 13 we have defined congruence in a number of different ways, dealing with various kinds of figures. The complete list looks like this:

(1) $\overline{AB} \cong \overline{CD}$ if the two segments have the same length, that is, if $AB = CD$.

(2) $\angle A \cong \angle B$ if the two angles have the same measure, that is, if $m\angle A = m\angle B$.

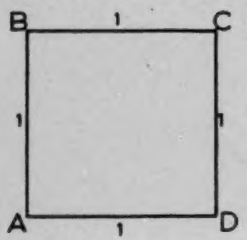
(3) $\triangle ABC \cong \triangle DEF$ if, under the correspondence $ABC \longleftrightarrow DEF$, every two corresponding sides are congruent and every two corresponding angles are congruent.

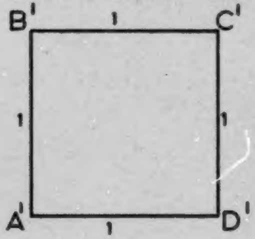
(4) Two circles are congruent if they have the same radius.

(5) Two circular arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are congruent if the circles that contain them are congruent and the two arcs have the same degree measure.

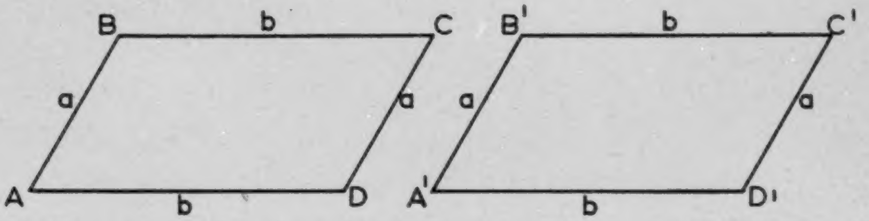
The intuitive idea of congruence is the same in all five of these cases. In each case, roughly speaking, two figures are congruent if one of them can be moved so as to coincide with the other; and in the case of triangles, a congruence is a way of moving the first figure so as to make it coincide with the second.

At the beginning of our study of congruence, the scheme used in Chapters 5 and 13 is the easiest and probably the best. It is a pity, however, to have five different special ways of describing the same basic idea in five special cases. And, in a way, it is a pity for this basic idea to be limited to these five special cases. For example, as a matter of common sense it is plain that two squares, each of edge 1, must be congruent in some valid sense:





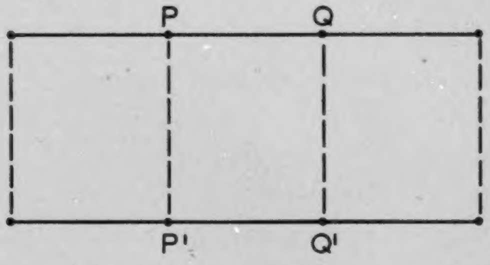
The same ought to be true for parallelograms, if corresponding sides and angles are congruent, like this:



It is plain, however, that none of our five special definitions of congruence applies to either of these cases.

In this appendix, we shall explain the idea of a rigid motion. This idea is defined in exactly the same way, regardless of the type of figure to which we happen to be applying it. We shall show that for segments, angles, triangles, circles and arcs it means exactly the same thing as congruence. Finally, we will show that the squares and parallelograms in the figures above can be made to coincide by rigid motion. Thus, first, the idea of congruence will be unified, and second, the range of its application will be extended.

Before we give the general definition of a rigid motion, let us look at some simple examples. Consider two opposite sides of a rectangle, like this:



The vertical sides are dotted, because we will not be especially concerned with them. For each point P, Q, ... and so on, of the top edge let us drop a perpendicular to the bottom edge; and let the foot of the perpendicular be P', Q' ... and so on.

Under this procedure, to each point of the top edge there corresponds exactly one point of the bottom edge. And conversely, to each point of the bottom edge there corresponds exactly one point of the top edge. We can't write down all of the matching pairs $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$, ... and so on, because there are infinitely many of them. We can, however, give a general rule, explaining what is to correspond to what; and in fact, this is what we have done. Usually we will write down a typical pair

$$P \longleftrightarrow P',$$

and explain the rule by which the pairs are to be formed.

Notice that the idea of a one-to-one correspondence is exactly the same in this case as it was when we were using it for triangles in Chapter 5. The only difference is that if we are matching up the vertices of two triangles, we can write down all of the matching pairs, because there are only three of them. ($ABC \longleftrightarrow DEF$ means that $A \longleftrightarrow D$, $B \longleftrightarrow E$ and $C \longleftrightarrow F$.) At present we are talking about exactly the same sort of things, only there are too many of them to write down.

It is very easy to check that if P and Q are any two points of the top edge, and P' and Q' are the corresponding points of the bottom edge, then

$$PQ = P'Q'.$$

This is true because the segments $\overline{PQ}$ and $\overline{P'Q'}$ are opposite sides of a rectangle. We express this fact by saying that the correspondence $P \longleftrightarrow P'$ preserves distances.

The correspondence that we have just set up is our first and simplest example of a rigid motion. To be exact:

Definition: Given two figures F and F', a rigid motion between F and F' is a one-to-one correspondence

$$P \longleftrightarrow P'$$

between the points of F and the points of F', preserving distances.

If the correspondence $P \longleftrightarrow P'$ is a rigid motion between F and F', then we shall write

$$F \approx F'.$$

This notation is like the notation $\Delta ABC \cong \Delta A'B'C'$ for congruences between triangles. We can read $F \approx F'$ as "F is isometric to F'." ("Isometric" means "equal measure.")

Problem Set VIII-1

1. Consider triangles ΔABC and $\Delta A'B'C'$, and suppose that
$$\Delta ABC \cong \Delta A'B'C'.$$
Let F be the set consisting of the vertices of the first triangle, and let F' be the set consisting of the vertices of the second triangle. Show that there is a rigid motion
$$F \approx F'.$$

2. Let F be the set consisting of the vertices of a square of edge 1, and let F' be the set consisting of the vertices of another square of edge 1, as in the figure at the beginning of this Appendix. Show that there is a rigid motion
$$F \approx F'.$$
(First you have to explain what corresponds to what, and second you have to verify that distances are preserved.)

3. Do the same for the vertices of the two parallelograms in the figure at the start of this Appendix.

4. Show that if F consists of three collinear points, and F' consists of three non-collinear points, then there is no rigid motion between F and F'. (What you will have to do is to assume that such a rigid motion exists, and then show that this assumption leads to a contradiction.)

5. Show that there is never a rigid motion between two segments of different lengths.

6. Show that there is never a rigid motion between a line and an angle. (Hint: Apply Problem 4.)

7. Show that given any two rays, there is a rigid motion between them. (Hint: Use the Ruler Placement Postulate.)

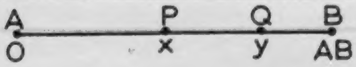
8. Show that there is never a rigid motion between two circles of different radius.

VIII-2. Rigid Motion of Segments.

Theorem VIII-1. If $AB = CD$, then there is a rigid motion $\overline{AB} \approx \overline{CD}$.

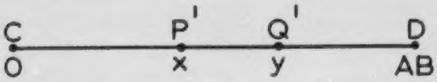
Proof: First we need to set up a correspondence $P \longleftrightarrow P'$ between $\overline{AB}$ and $\overline{CD}$. Then we need to check that distances are preserved.

By the Ruler Postulate, the points of the line $\overleftrightarrow{AB}$ can be given coordinates in such a way that the distance between any two points is the absolute value of the difference of the coordinates. And by the Ruler Placement Postulate, this can be done in such a way that A has coordinate zero and B the positive coordinate AB.



In the figure, we have shown typical points P, Q with their coordinates x and y.

In the same way, the points of $\overleftrightarrow{CD}$ can be given coordinates:



Notice that D has the coordinate AB, because $CD = AB$.

It is now plain what rule we should use to set up the correspondence

$$P \longleftrightarrow P'$$

between the points of $\overline{AB}$ and the points of $\overline{CD}$. The rule is that P corresponds to P' if P and P' have the same coordinate. (In particular, $A \longleftrightarrow C$ because A and C have coordinate zero, and $B \longleftrightarrow D$ because B and D have coordinate AB.)

It is easy to see that this correspondence is a rigid motion. If $P \longleftrightarrow P'$ and $Q \longleftrightarrow Q'$, and the coordinates are x and y, as in the figure, then $PQ = P'Q'$, because

$$PQ = |y - x| = P'Q'.$$

We therefore have a rigid motion

$$\overline{AB} \approx \overline{CD},$$

and the theorem is proved.

Notice that this rigid motion between the two segments is completely described if we explain how the end-points are to be matched up. We therefore will call it the rigid motion induced by the correspondence

$$A \longleftrightarrow C$$
$$B \longleftrightarrow D.$$

Theorem VIII-2. If there is a rigid motion $\overline{AB} \approx \overline{CD}$ between two segments, then $AB = CD$.

The proof is easy. (This theorem was Problem 5 in the previous Problem Set.)

Problem Set VIII-2

1. Show that there is another rigid motion between the congruent segments $\overline{AB}$ and $\overline{CD}$, induced by the correspondence

$$A \longleftrightarrow D$$
$$B \longleftrightarrow C.$$

2. Show that there are two rigid motions between a segment and itself. (One of these, of course, is the identity correspondence $P \longleftrightarrow P'$, under which every point corresponds to itself; this is a rigid motion because $PQ = PQ$ for every P and Q.)

VIII-3. Rigid Motion of Rays, Angles and Triangles.

Theorem VIII-3. Given any two rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$, there is a rigid motion

$$\overrightarrow{AB} \approx \overrightarrow{CD}.$$

The proof of this theorem is quite similar to that of Theorem VIII-1, and the details are left to the reader.

Theorem VIII-4. If $\angle ABC \cong \angle DEF$, then there is a rigid motion

$$\angle ABC \approx \angle DEF$$

between these two angles.

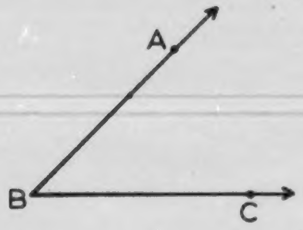
Proof: We know that there are rigid motions

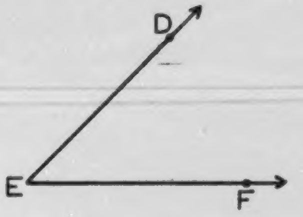
$$\overrightarrow{BA} \approx \overrightarrow{ED}$$

and

$$\overrightarrow{BC} \approx \overrightarrow{EF}$$

between the rays which form the sides of the two angles.



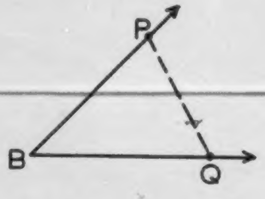


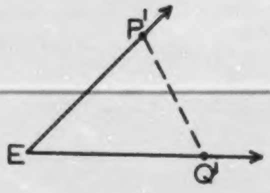
Let us agree that two points P and P' (or Q and Q') are to correspond to one another if they correspond under one of these two rigid motions. This gives us a one-to-one correspondence between the two angles. What we need to show is that this correspondence preserves distances.

Suppose that we have given two points P, Q of $\angle ABC$ and the corresponding points P', Q' of $\angle DEF$. If P and Q are on the same side of $\angle ABC$, then obviously

$$P'Q' = PQ,$$

because distances are preserved on each of the rays that form $\angle ABC$. Suppose, then, that P and Q are on different sides of $\angle ABC$, so that P' and Q' are on different sides of $\angle DEF$, like this:





By the S.A.S. Postulate, we have

$$\Delta\ PBQ \cong \Delta\ P'EQ'.$$

Therefore $PQ = P'Q'$, which was to be proved.

Next, we need to prove the analogous theorem for triangles:

Theorem VIII-5. If

$$\Delta\ ABC \cong \Delta\ A'B'C',$$

then there is a rigid motion

$$\Delta\ ABC \approx \Delta\ A'B'C',$$

under which A, B and C correspond to A', B' and C'.

Proof: First we shall set up a one-to-one correspondence between the points of Δ ABC and the points of Δ A'B'C'. We have given a one-to-one correspondence

$$ABC \longleftrightarrow A'B'C'$$

for the vertices. By Theorem VIII-1 this gives us the induced rigid motions

$$\overline{AB} \approx \overline{A'B'},$$
$$\overline{AC} \approx \overline{A'C'}$$

and

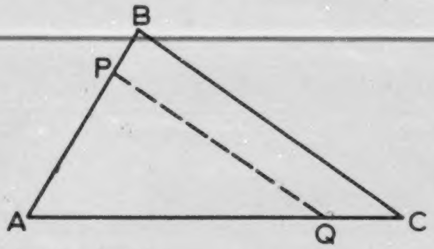
$$\overline{BC} \approx \overline{B'C'}$$

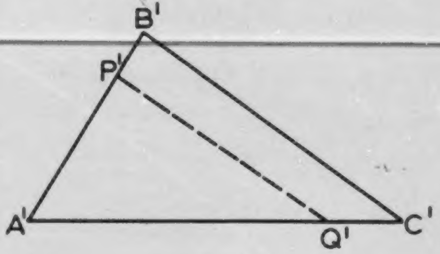
between the sides of the triangles. These three rigid motions, taken together, give us a one-to-one correspondence $P \longleftrightarrow P'$ between the points of the two triangles. We need to show that this correspondence preserves distances.

If P and Q are on the same side of the triangle, then we know already that

$$P'Q' = PQ.$$

Suppose, then, that P and Q are on different sides, say, $\overline{AB}$ and $\overline{AC}$, like this:





We know that

$$AP = A'P',$$

because $\overline{AB} \approx \overline{A'B'}$ is a rigid motion. For the same reason,

$$AQ = A'Q',$$

and $\angle A \cong \angle A'$, because $\Delta\ ABC \cong \Delta\ A'B'C'$. By the S.A.S. Postulate,

$$\Delta\ PAQ \cong \Delta\ P'A'Q'.$$

Therefore,

$$PQ = P'Q',$$

which was to be proved.

Notice that while the figure does not show the case $P = B$, the proof takes care of this case. The proof is more important than the figure, anyway.

Problem Set VIII-3

1. Let

$$ABC \longrightarrow A'B'C'$$

be a rigid motion, and suppose that A, B and C are collinear. Show that if B is between A and C, then B' is between A' and C'.

2. Given a rigid motion

$$F \approx F'.$$

Let A and B be points of F, and suppose that F contains the segment $\overline{AB}$. Show that F' contains the segment $\overline{A'B'}$.

3. Given a rigid motion

$$F \approx F'.$$

Show that if F is convex, then so also is F'.

4. Given a rigid motion

$$F \approx F'.$$

Show that if F is a segment, then so also is F'.

5. Given a rigid motion $F \approx F'$. Show that if F is a ray, then so also is F'.

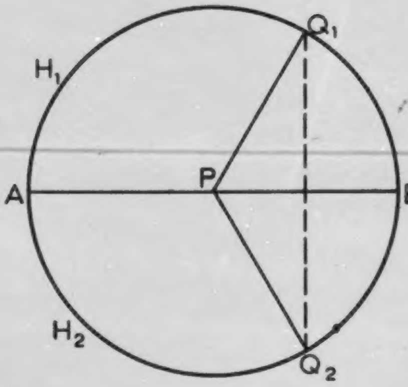
6. Show that there is no rigid motion between a segment and a circular arc (no matter how short both of them may be).

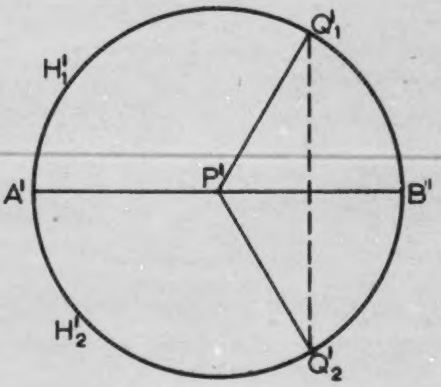
VIII-4. Rigid Motion of Circles and Arcs.

Theorem VIII-6. Let C and C' be circles of the same radius r. Then there is a rigid motion

$$C \approx C'$$

between C and C'.





Proof: Let the centers of the circles be P and P'. Let $\overline{AB}$ be a diameter of the first circle, and let $\overline{A'B'}$ be a diameter of the second. Let H_1 and H_2 be the half-planes determined by the line $\overleftrightarrow{AB}$; and let H'_1 and H'_2 be the half-planes determined by the line $\overleftrightarrow{A'B'}$.

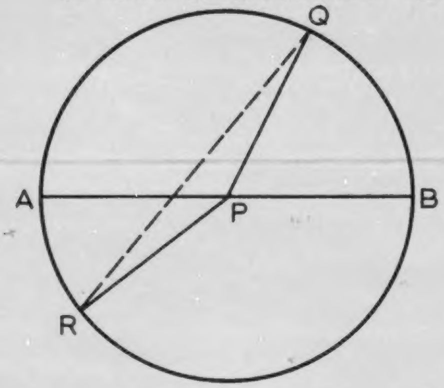
We can now set up our one-to-one correspondence $Q \longleftrightarrow Q'$, in the following way: (1) Let A' and B' correspond to A and B, respectively. (2) If Q_1 is a point of C, lying in H_1, let Q'_1 be the point of C', lying in H'_1, such that

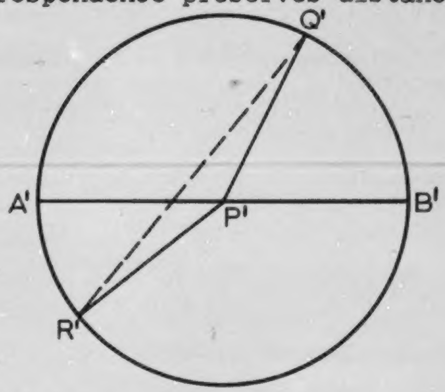
$$\angle Q'_1P'B' \cong \angle Q_1PB.$$

(3) If Q_2 is a point of C, lying in H_2, let Q'_2 be the point of C_2, lying in H'_2, such that

$$\angle Q'_2P'B' \cong \angle Q_2PB.$$

We need to check that this correspondence preserves distances.





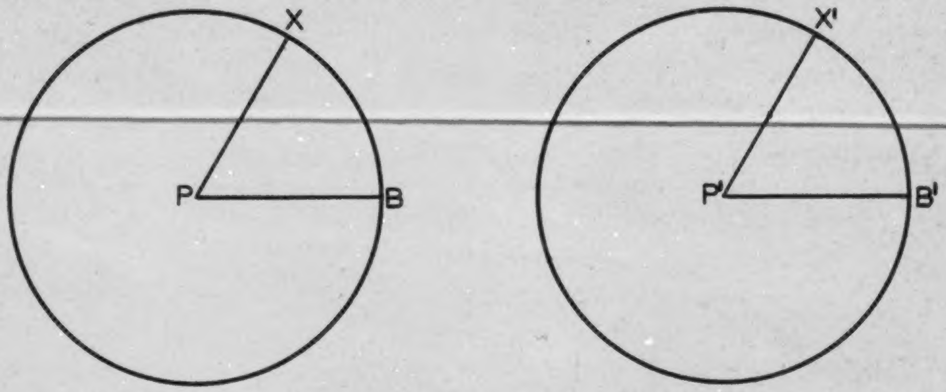
Thus, for every two points Q, R of C, we must have

$$Q'R' = QR.$$

If Q and R are the end-points of a diameter, then so are Q' and R', and $Q'R' = QR = 2r$. Otherwise, we always have $\triangle QPR \cong \triangle Q'P'R'$, so that $Q'R' = QR$. (Proof? There are two cases to consider, according as B is in the interior or the exterior of $\angle QPR$.)

You should prove the following two theorems for yourself. They are not hard, once we have gone this far.

Theorem VIII-7. Let C and C' be circles with the same radius, as in Theorem VIII-6. Let $\angle XPB$ and $\angle X'P'B'$ be congruent central angles of C and C', respectively.



Then a rigid motion $C \approx C'$ can be chosen in such a way that $B \longleftrightarrow B'$, $X \longleftrightarrow X'$, and $\overset{\frown}{BX} \approx \overset{\frown}{B'X'}$.

Theorem VIII-8. Given any two congruent arcs, there is a rigid motion between them. The proof is left to the reader.

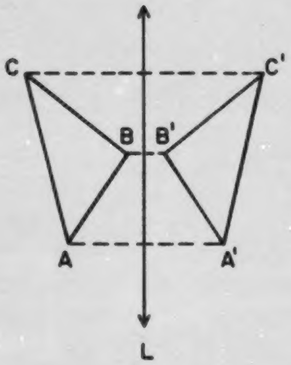
VIII-5. Reflections.

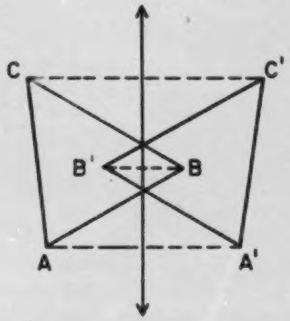
The definition of rigid motion given in Section VIII-1 is a perfectly good mathematical definition, but we might claim that from an intuitive viewpoint it does not convey any idea of "motion". We will devote this section to showing how a plane figure can be "moved" into coincidence with any isometric figure in the same plane.

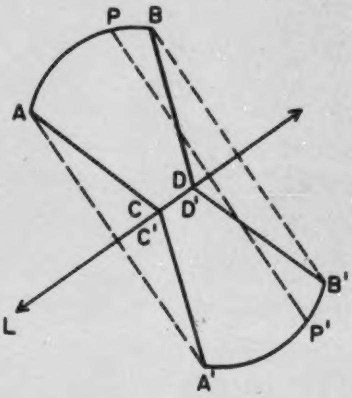
Throughout this section all figures will be considered as lying in a fixed plane.

Definitions. A one-to-one correspondence between two figures is a reflection if there is a line L, such that for any pair of corresponding points P and P', either (1) P = P' and lies on L or (2) L is the perpendicular bisector of $\overline{PP'}$. L is called the axis of reflection, and each figure is said to be the reflection, or the image, of the other figure in L.

In the pictures below are shown some examples of reflections of simple figures.

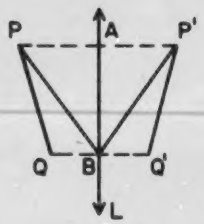




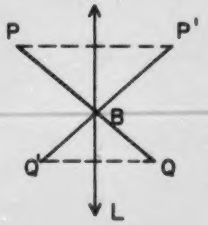


Theorem VIII-9. A reflection is a rigid motion.

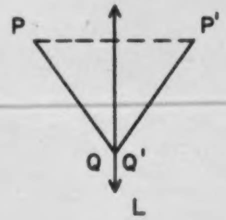
Proof: We must show that if P and Q are any two points, and P' and Q' their images in a line L, then PQ = P'Q'. There are four cases to consider.



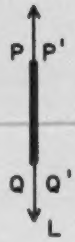
Case (1)



Case (2)



Case (3)



Case (4)

Case 1. P and Q are on the same side of L. Let $\overline{PP'}$ intersect L at A and $\overline{QQ'}$ intersect L at B. By the definition of reflection $\overline{PP'} \perp L$ and $PA = P'A$, and $\overline{QQ'} \perp L$ and $QB = Q'B$. Hence $\triangle PAB \cong \triangle P'AB$, and $PB = P'B$, $\angle PBA \cong \angle P'BA$. By subtraction, $\angle PBQ \cong \angle PBQ'$. We then have (by S.A.S.) $\triangle PBQ \cong \triangle P'BQ'$, and so $PQ = P'Q'$.

Case 2. The proof is the same, except that in proving $\angle PBQ \cong \angle PBQ'$ we add angle measures instead of subtracting.

Case 3. Q is on L. Then $Q = Q'$ and $PQ = P'Q'$ since Q is on the perpendicular bisector of $\overline{PP'}$. The case P on L and Q not on L is just the same.

Case 4. P and Q both on L. Since $P = P'$ and $Q = Q'$ we certainly have $PQ = P'Q'$.

Starting with a figure F we can reflect it in some line to get a figure F_1, F_1 can be reflected in some line to get a figure F_2, and so on. If we end up with a figure F' after n such steps we shall say that F has been carried into F' by a chain of n reflections.

Corollary VIII-9-1. A chain of reflections carrying F into F' determines a rigid motion between F and F'.

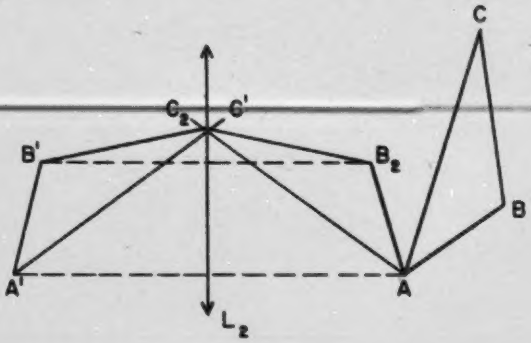
Coming back to our opening discussion in this section, a reflection can be thought of as a physical motion, obtained by rotating the whole plane through 180° about the axis of reflection. The above corollary says that a certain type of rigid motion, namely, those obtainable as a chain of reflections, can be given a physical interpretation. What we shall now show is that every rigid motion is of this type.

The proof will be given in two stages, the first stage involving only a very simple figure. For convenience we will use the notation $F \mid F'$ if F and F' are reflections of each other in some axis.

Theorem VIII-10. Let A, B, C, A', B', C' be six points such that AB = A'B', AC = A'C', BC = B'C'. Then there is a chain of at most three reflections that carries A, B, C into A', B', C'.

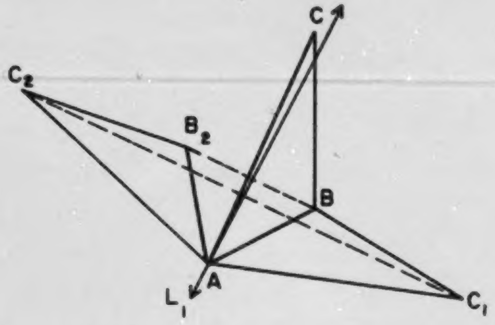
Proof:

Step 1.



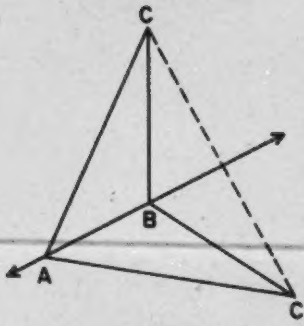
Let L_2 be the perpendicular bisector of $\overline{AA'}$, and let B_2 and C_2 be the reflections of B' and C' in L_2. Then A, B_2,C_2 | A',B',C'.

Step 2.



Let L_1 be the perpendicular bisector of $\overline{BB_2}$. Since AB = A'B' and since by Theorem VIII-9, A'B' = AB_2, it follows that AB = AB_2. Therefore A lies on L_1 and is its own image in the reflection in L. Thus, the image of A, B_2, C_2 in L_1 is A, B, C_1.

Step 3.



By arguments similar to the one above we see that $AC = AC_1$ and $BC = BC_1$. Hence, $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{CC_1}$, and the image A, B, C_1 in $\overleftrightarrow{AB}$ is A, B, C.

We thus have,

$$A,B,C \mid A,B,C_1 \mid A,B_2,C_2 \mid A',B',C',$$

as was desired.

Any one or two of the three steps may be unnecessary if the pair of points we are working on (A in step 1, B in step 2, C in step 3) happen to coincide.

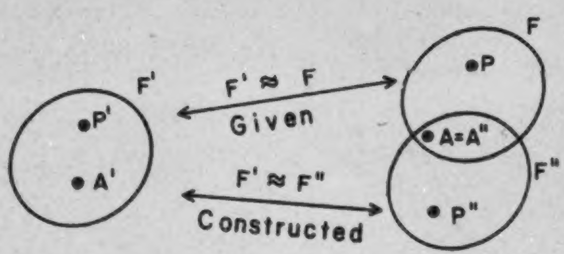
We are now ready for the final stage of the proof.

Theorem VIII-11. Any rigid motion is the result of a chain of at most three reflections.

Proof: We are given a rigid motion $F \approx F'$. Let A, B, C be three non-collinear points in F, and A', B', C' the corresponding points in F'.

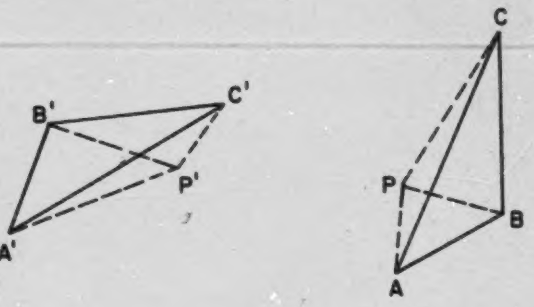
(If all points of F are collinear a separate, but simpler, proof is needed. The details of this are left to the student.)

By Theorem VIII-10 we can pass from A', B', C' to A, B, C by a chain of at most three reflections. By Corollary VIII-9-1 this chain determines a rigid motion $F' \approx F''$, and by the construction of the reflections we have $A'' = A$, $B'' = B$ and $C'' = C$. Schematically the situation is something like this:



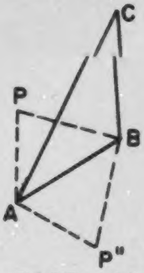
We shall show that for every point P of F we have $P'' = P$. This will show that F'' coincides with F, and that the given rigid motion $F \approx F'$ is identical with the one determined by the chain of reflections.

Let us consider, then, any point P of F, its corresponding point P' in F' determined by the rigid motion $F \approx F'$, and the point P'' in F'' determined from P' by the chain of reflections. We recall that $A'' = A$, $B'' = B$, $C'' = C$.



Since all our relationships are rigid motions we have $AP'' = A'P' = AP$. Similarly, $BP'' = BP$ and $CP'' = CP$. From the first two of these, and $AB = AB$, we get that $\Delta ABP \cong \Delta ABP''$, and so $\angle BAP = \angle BAP''$. If P and P'' are on the same side of $\overleftrightarrow{AB}$ then by the Angle Construction Postulate $\overrightarrow{AP} = \overrightarrow{AP''}$, and since $AP = AP''$ it follows from the Point Plotting Theorem that $P = P''$, which is what we wanted to prove.

Suppose then that P and P'' lie on opposite sides of $\overleftrightarrow{AB}$.

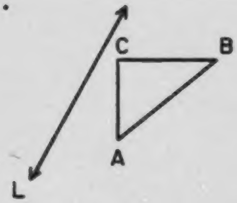


Since PA = P''A and PB = P''B it follows that A and B lie on the perpendicular bisector of $\overline{PP''}$. Since PC = P''C, C also lies on this line, contrary to the choice of A, B and C as non-collinear. Hence, this case does not arise, and we are left with P = P'', thus proving the theorem.

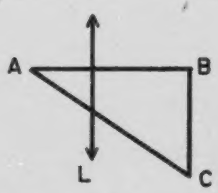
Problem Set VIII-5

1. In each of the following construct, with any instruments you find convenient, the image of the given figure in the line L.

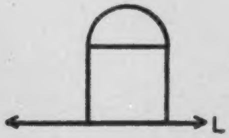
a.



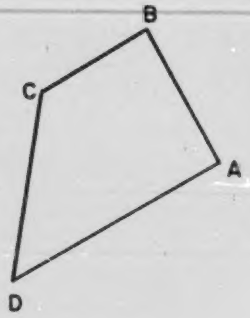
b.

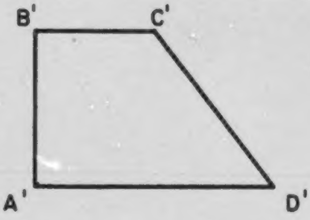


c.

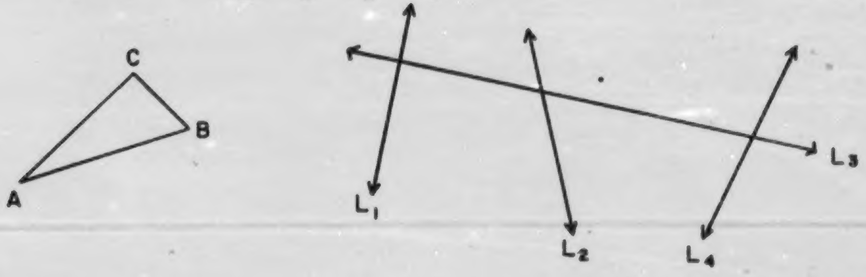


2. Find a chain of three or fewer reflections that will carry ABCD into A'B'C'D'.





3. a. Carry Δ ABC through the chain of four reflections in the axes L_1, L_2, L_3, L_4.



b. Find a shorter chain that will give the same rigid motion.

Definitions: A figure is *symmetric* if it is its own image in some axis. Such an axis is called an *axis of symmetry* of the figure.

4. Show that an isosceles triangle is symmetric. What is the axis?

5. A figure may have more than one axis of symmetry. How many do each of the following figures have?

   a. A rhombus.

   b. A rectangle.

   c. A square.

   d. An equilateral triangle.

   e. A circle.

6. The rigid motion defined by a chain of two reflections in parallel axes has the property that if $P \longleftrightarrow P'$ then $\overline{PP'}$ has a fixed length (twice the distance between the axes) and direction (perpendicular to the axes). Prove this. Such a motion is called a *translation*.

7. The rigid motion defined by a chain of two reflections in axes which intersect at Q has the property that if $P \longleftrightarrow P'$ then $\angle PQP'$ has a fixed measure (twice the measure of the acute angle between the axes). Prove this.
Such a motion is called a rotation about Q.

8. Show how by using the results of Problems 6 and 7 the Fundamental Theorem VIII-11 can be restated in the following form:
Any rigid motion in a plane is either a reflection, a translation, a rotation, a translation followed by a reflection, or a rotation followed by a reflection.

Appendix IX

PROOF OF THE TWO-CIRCLE THEOREM

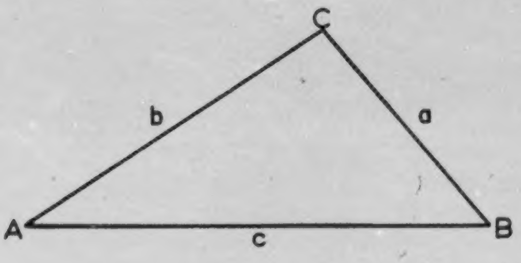
The validity of the Two Circle Theorem, stated in Chapter 14, rests on the existence of a certain triangle, and the proof is easier to follow if we establish this first.

Triangle Existence Theorem. If a, b, c are positive numbers, each of which is less than the sum of the other two, then there is a triangle whose sides have lengths a, b, c.

Proof: The hard part of the proof is algebraic rather than geometric. First, let us suppose, as a matter of notation, that the three numbers a, b, c are written in order of magnitude, so that

$$a \leq b \leq c.$$

Let us start with a segment $\overline{AB}$, with $AB = c$. Our problem is to find a triangle ΔABC, with $BC = a$ and $AC = b$, like this:



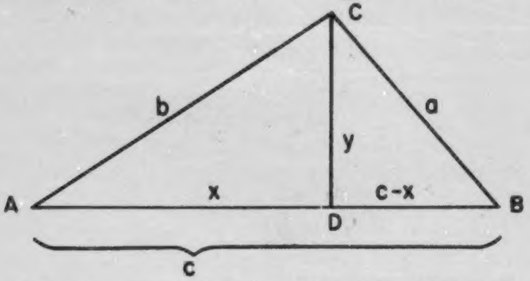
In a sense we are going to tackle this problem backwards. That is, we are going to start off by assuming that there _is_ such a triangle. On the basis of this assumption, we will find out exactly where the third vertex C must be. This procedure in itself will not, of course, prove that the above statement is true, because we started by assuming the very thing that we are supposed to be proving. But once we have found the exact location of the points that _might_ work, it will be very easy to check that these points really _do_ work. (Of course, there are two possible places for C, on the two sides of the line $\overleftrightarrow{AB}$.)

(This procedure is just what we use in solving equations. To solve $3x - 7 = x + 3$ we first assume that there is an x which satisfies this equation. For this x we find successively that

$$3x = x + 10,$$
$$2x = 10,$$
$$x = 5.$$

Then we reverse our steps and show that 5 actually does satisfy the given equation.)

Suppose, then, that there is a triangle ΔABC of the sort that we are looking for. Let us drop a perpendicular from C to $\overleftrightarrow{AB}$, and let D be the foot of the perpendicular. Then D is between A and B, because $AD < b \leq c$ and $BD < a \leq c$.



Let $y = CD$, and let $x = AD$, as in the figure. Then $DB = c - x$, as indicated. We want to find out what x and y are equal to, in terms of a, b and c.

By the Pythagorean Theorem, we have

$$(1) \qquad x^2 + y^2 = b^2$$

and

$$(2) \qquad y^2 + (c - x)^2 = a^2.$$

Therefore

$$y^2 = b^2 - x^2$$

and

$$y^2 = a^2 - (c - x)^2.$$

Equating the two expressions for y^2 we see that

$$b^2 - x^2 = a^2 - (c - x)^2,$$

$$b^2 - x^2 = a^2 - c^2 + 2cx - x^2,$$

$$2cx = b^2 + c^2 - a^2,$$

and

$$(3) \qquad x = \frac{b^2 + c^2 - a^2}{2c}.$$

What we have found, so far, is that if x and y satisfy (1) and (2), then x satisfies (3). We shall check, conversely, that if x and y satisfy (1) and (3), then x and y also satisfy (2). For if (1) and (3) hold, then we have from (1) that

$$y^2 = b^2 - x^2.$$

Adding $(c - x)^2$ to both sides we get

$$\begin{aligned} y^2 + (c - x)^2 &= (b^2 - x^2) + (c - x)^2 \\ &= b^2 - x^2 + c^2 - 2cx + x^2 \\ &= b^2 + c^2 - 2cx. \end{aligned}$$

Substituting for x from (3) gives

$$\begin{aligned} y^2 + (c - x)^2 &= b^2 + c^2 - (b^2 + c^2 - a^2) \\ &= a^2, \end{aligned}$$

so that (2) holds.

Now that we know what triangle to look for, let us start all over again. We have three positive numbers, a, b, c. Each of them is less than the sum of the other two, and $a \leq b \leq c$. Let

$$x = \frac{b^2 + c^2 - a^2}{2c}.$$

Then $x > 0$, because $b^2 \geq a^2$ and $c^2 > 0$. We want to set

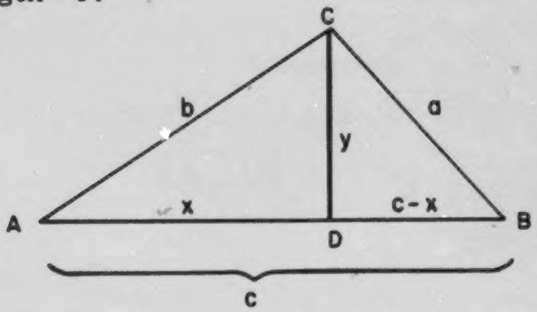
$$y = \sqrt{b^2 - x^2},$$

so that $x^2 + y^2 = b^2$, but to do this we must first make sure that $x < b$, that is, that $b - x > 0$. We have

$$\begin{aligned} b - x &= b - \frac{b^2 + c^2 - a^2}{2c} \\ &= \frac{2bc - b^2 - c^2 + a^2}{2c} \\ &= \frac{a^2 - (c^2 - 2bc + b^2)}{2c} \\ &= \frac{a^2 - (c - b)^2}{2c} \end{aligned}$$

Now we are given that $c < a + b$. Hence, $c - b < a$ and so $(c - b)^2 < a^2$. It follows from the equation above that $b - x > 0$, or $x < b$.

We are now ready to construct our triangle. Let $\overline{AB}$ be a segment of length c.



Let D be a point on $\overline{AB}$ such that $AD = x = \frac{b^2 + c^2 - a^2}{2c}$. Such a point exists since we know $x < b \leq c$. Let C be a point on the perpendicular to $\overline{AB}$ through D, such that

$$DC = y = \sqrt{b^2 - x^2}.$$

Then

$$AC^2 = x^2 + y^2 = b^2,$$

and

$$BC^2 = y^2 + (c - x)^2 = a^2.$$

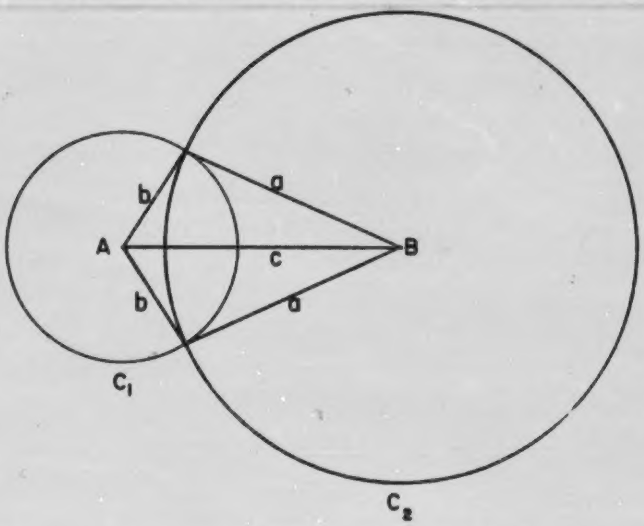
Therefore $AC = b$ and $BC = a$, which is what we wanted.

The proof of the Two Circle Theorem is now fairly easy.

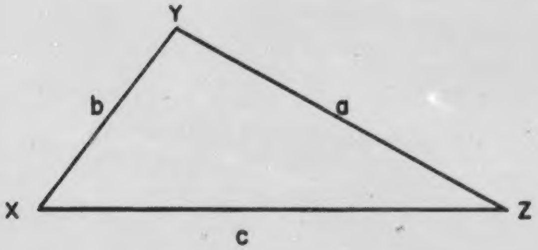
Theorem 14-5. (The Two Circle Theorem.)

If two circles have radii a and b, and if c is the distance between their centers, then the two circles intersect in two points, one on each side of the line of centers, provided each one of a, b, c is less than the sum of the other two.

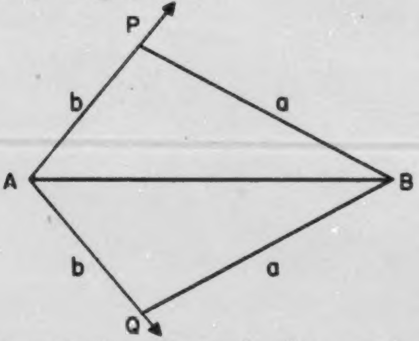
Proof: Let C_1, the circle with radius b, have center A, and C_2, the circle with radius a have center B. Then AB = C.



We know by the Triangle Existence Theorem that there is a triangle Δ XYZ whose sides have lengths a, b, and c, like this:



Using the S.A.S. Postulate, we are going to copy this triangle on each side of the line $\overleftrightarrow{AB}$, in the following way. On each side of $\overleftrightarrow{AB}$ we take a ray starting at A, in such a way that the angles formed are congruent to angle X.



On these rays we take points P and Q, such that $AP = AQ = b$. Therefore circle C_1 passes through P and Q. By the S.A.S. Postulate,

$$\Delta\ APB \cong \Delta\ XYZ \cong \Delta\ AQB.$$

Therefore $PB = a = QB$, and hence circle C_2 passes through P and Q.

This shows that P and Q are at least part of the intersection of C_1 and C_2. To show that they are the intersection we must prove that no third point, R, can lie on both C_1 and C_2. If there were such a point R we would have, by the S.S.S. Theorem

$$\Delta\ ABR \cong \Delta\ ABP, \text{ and so, } m\angle\ BAR = m\angle\ BAP.$$

But in the given plane there are only two such angles, one on each side of $\overleftrightarrow{AB}$, and hence, either $\overrightarrow{AR} = \overrightarrow{AP}$ or $\overrightarrow{AR} = \overrightarrow{AQ}$. Since $AR = AP = AQ = b$ this means that either $R = P$ or $R = Q$, and so there can be no third point on both C_1 and C_2.

Appendix X

TRIGONOMETRY

X-1. Trigonometric Ratios.

The elementary study of trigonometry is based on the following theorem.

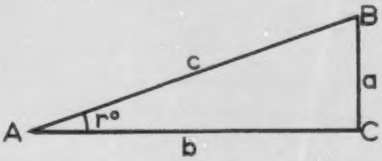
Theorem X-1. If an acute angle of one right triangle is congruent to an acute angle of another right triangle, then the two triangles are similar.

Proof: In Δ ABC and Δ A'B'C' let $\angle C$ and $\angle C'$ be right angles and let $m\angle A = m\angle A'$. Then $\Delta ABC \sim \Delta A'B'C'$ by A.A. Similarity Corollary 12-3-1.

We apply this theorem as follows: Let r be any number between 0 and 90, and let Δ ABC be a right triangle with $m\angle C = 90$ and $m\angle A = r$. For convenience set

$$AB = c, \quad AC = b, \quad BC = a.$$

(The Pythagorean Theorem then tells us that $c^2 = a^2 + b^2$.)



If we consider another such triangle Δ A'B'C' with $m\angle C' = 90$ and $m\angle A' = r$, we get three corresponding numbers a', b', c', which would generally be different from a, b, c. However, we always have

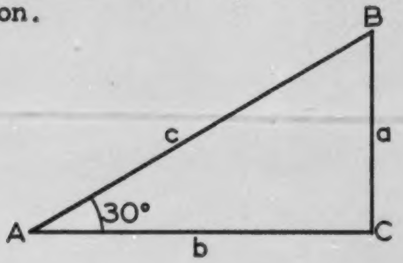
$$\frac{a'}{c'} = \frac{a}{c}.$$

To see this, note that it follows from Theorem X-1 that

$$\frac{a'}{a} = \frac{c'}{c}.$$

If we multiply both sides of this equation by $\frac{a}{c'}$ we get the desired result.

Thus the ratio $\frac{a}{c}$ does not depend on the particular triangle we use, but only on the measure r of the acute angle. The value of this ratio is called the sine of r°, written $\sin r^\circ$ for short. The reason we specify that we are using degree measure is that in more advanced aspects of trigonometry a different measure of angle, radian measure, is common.



Let us see what we can say about $\sin 30^\circ$. We know from Theorem 11-9 that in this case if $c = 1$, then $a = \frac{1}{2}$. Hence, $\sin 30^\circ = \frac{a}{c} = \frac{1}{2}$.

It is evident that the ratio $\frac{b}{c}$ can be treated in the same way as $\frac{a}{c}$. The ratio $\frac{b}{c}$ is called the cosine of r°, written $\cos r^\circ$. From the Pythagorean Theorem we see that if $a = \frac{1}{2}$ and $c = 1$, then $b = \frac{\sqrt{3}}{2}$. Hence, $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

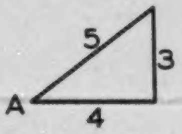
Of the four other possible ratios of the three sides of the triangle, we shall use only one, $\frac{a}{b}$. This is called the tangent of r°, written $\tan r^\circ$. We see that $\tan 30^\circ = \frac{1}{\sqrt{3}}$. (This use of the word "tangent" has only an unimportant historical connection with its use with relation to a line and a circle.)

These three quantities are called trigonometric ratios.

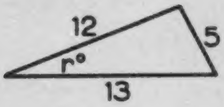
Problem Set X-1

1. In each of the following give the required information in terms of the indicated lengths of the sides.

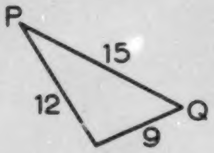
   a. $\sin A = ?$, $\cos A = ?$, $\tan A = ?$.



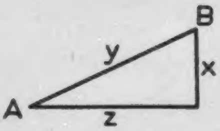
   b. $\sin r^\circ = ?$, $\cos r^\circ = ?$, $\tan r^\circ = ?$.



   c. $\sin P = ?$, $\cos P = ?$, $\tan Q = ?$.

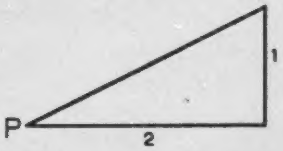


   d. $\sin A = ?$, $\sin B = ?$,
   $\tan A = ?$, $\tan B = ?$.

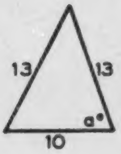


2. In each of the following find the correct numerical value for x.

   a. $\cos P = x$.



   b. $\tan a^\circ = x$.



3. Find: sin 60°, cos 60°, tan 60°.

4. Find: sin 45°, cos 45°, tan 45°.

5. By making careful drawings with ruler and protractor determine by measuring

   a. sin 20°, cos 20°, tan 20°;

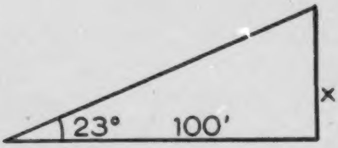
   b. sin 53°, cos 53°, tan 53°.

X-2. Trigonometric Tables and Applications.

Although the trigonometric ratios can be computed exactly for a few angles, such as 30°, 60° and 45°, in most cases we have to be content with approximate values. These can be worked out by various advanced methods, and at the end of this Appendix we give a table of the values of the three trigonometric ratios correct to three decimal places.

Having a "trig table", and a device for measuring angles, such as a surveyor's transit (or strings and a protractor) one can solve various practical problems.

Example X-1. From a point 100 feet from the base of a flag pole the angle between the horizontal and a line to the top of the pole is found to be 23°. Let x be the height of the pole. Then



$$\frac{x}{100} = \tan 23^\circ = .425.$$

Hence, $x = 42.5$ feet. An angle like the one used in this example is frequently called the angle of elevation of the object.

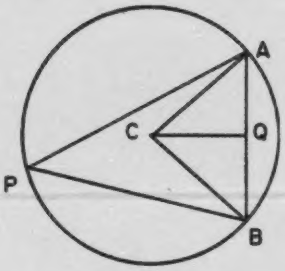
Example X-2. In a circle of radius 8 cm. a chord $\overline{AB}$ has length 10 cm. What is the measure of an angle inscribed in the major arc $\widehat{AB}$? We have $AC = 8$, $AQ = \frac{1}{2} \cdot 10 = 5$. Hence,

$\sin \angle ACQ = \frac{5}{8} = .625$,

$m\angle ACQ = 39^\circ$,

$m(\text{minor arc } \widehat{AB}) = m\angle ACB =$
$2(m\angle ACQ) = 78^\circ$.

Hence, $m\angle APB = \frac{1}{2}m(\text{arc } \widehat{AB}) =$
39° to the nearest degree.



Problem Set X-2.

1. From the table find: $\sin 17^\circ$, $\cos 46^\circ$, $\tan 82^\circ$, $\cos 33^\circ$, $\sin 60^\circ$. Does the last value agree with the one found in Problem 3 of Set X-1?

2. From the table find x to the nearest degree in each of the following cases:

   $\cos x = .731$, $\sin x = .390$, $\tan x = .300$

   $\sin x = .413$, $\tan x = 2$, $\cos x = \frac{1}{3}$.

3. A hiker climbs for a half mile up a slope whose inclination is 17°. How much altitude does he gain?

4. When a six-foot pole casts a four-foot shadow what is the angle of elevation of the sun?

5. An isosceles triangle has a base of 6 inches and an opposite angle of 30°. Find:

   a. The altitude of the triangle.

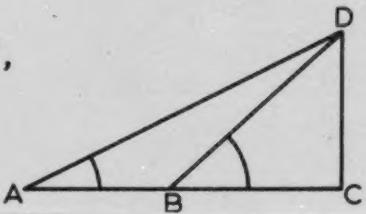
   b. The lengths of the altitudes to the equal sides.

   c. The angles these altitudes make with the base.

   d. The point of intersection of the altitudes.

6. A regular decagon (10 sides) is inscribed in a circle of radius 12. Find the length of a side, the apothem, and the area of the decagon.

7. Given, $m\angle A = 26^\circ$, $m\angle CBD = 42^\circ$, $BC = 50$; find AD and AB.



X-3. Relations Among the Trigonometric Ratios.

Theorem X-2. For any acute $\angle A$, $\sin A < 1$, $\cos A < 1$.

Proof: In the right triangle Δ ABC of Section X-1, $a < c$ and $b < c$. Dividing each of these inequalities by c gives

$$\frac{a}{c} < 1, \quad \frac{b}{c} < 1,$$

which is what we wanted to prove.

Theorem X-3. For any acute angle A,

$$\frac{\sin A}{\cos A} = \tan A, \quad \text{and} \quad (\sin A)^2 + (\cos A)^2 = 1.$$

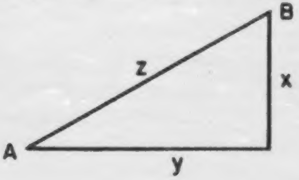
Proof:

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} = \tan A.$$

$$(\sin A)^2 + (\cos A)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

$$= \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1.$$

Theorem X-4. If $\angle A$ and $\angle B$ are complementary acute angles, then $\sin A = \cos B$, $\cos A = \sin B$, and $\tan A = \frac{1}{\tan B}$.



Proof: In the notation of the figure we have

$$\sin A = \frac{x}{z} = \cos B,$$

$$\cos A = \frac{y}{z} = \sin B,$$

$$\tan A = \frac{x}{y} = \frac{1}{\frac{y}{x}} = \frac{1}{\tan B}.$$

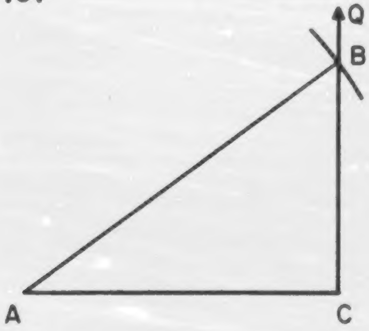
Problem Set X-3

Do the following problems without using the tables.

1. If $\sin A = \frac{1}{3}$ what is the value of $\cos A$? What is the value of $\tan A$? (Use Theorem X-3.)

2. With ruler and compass construct $\angle A$, if possible, in each of the following. You are allowed to use the results of earlier parts to simplify later ones.

   a. $\cos A = .8$.



Solution: Take $\overline{AC}$ any convenient segment and construct $\overrightarrow{CQ} \perp \overline{AC}$. With center A and radius $\frac{AC}{.8}$ construct an arc intersecting $\overrightarrow{CQ}$ at B. Then $\cos(\angle BAC) = .8$.

b. $\cos A = \frac{2}{3}$.

c. $\cos A = \frac{3}{2}$.

d. $\sin A = .8$.

e. $\sin A = .7$.

f. $\tan A = \frac{2}{3}$.

g. $\tan A = \frac{3}{2}$.

Table of Trigonometric Ratios

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
0	0.000	1.000	0.000				
1	.017	1.000	.017	46	0.719	0.695	1.036
2	.035	0.999	.035	47	.731	.682	1.072
3	.052	.999	.052	48	.743	.669	1.111
4	.070	.998	.070	49	.755	.656	1.150
5	.087	.996	.087	50	.766	.643	1.192
6	.105	.995	.105	51	.777	.629	1.235
7	.122	.993	.123	52	.788	.616	1.280
8	.139	.990	.141	53	.799	.602	1.327
9	.156	.988	.158	54	.809	.588	1.376
10	.174	.985	.176	55	.819	.574	1.428
11	.191	.982	.194	56	.829	.559	1.483
12	.208	.978	.213	57	.839	.545	1.540
13	.225	.974	.231	58	.848	.530	1.600
14	.242	.970	.249	59	.857	.515	1.664
15	.259	.966	.268	60	.866	.500	1.732
16	.276	.961	.287	61	.875	.485	1.804
17	.292	.956	.306	62	.883	.469	1.881
18	.309	.951	.325	63	.891	.454	1.963
19	.326	.946	.344	64	.899	.438	2.050
20	.342	.940	.364	65	.906	.423	2.145
21	.358	.934	.384	66	.914	.407	2.246
22	.375	.927	.404	67	.921	.391	2.356
23	.391	.921	.424	68	.927	.375	2.475
24	.407	.914	.445	69	.934	.358	2.605
25	.423	.906	.466	70	.940	.342	2.747
26	.438	.899	.488	71	.946	.326	2.904
27	.454	.891	.510	72	.951	.309	3.078
28	.469	.883	.532	73	.956	.292	3.271
29	.485	.875	.554	74	.961	.276	3.487
30	.500	.866	.577	75	.966	.259	3.732
31	.515	.857	.601	76	.970	.242	4.011
32	.530	.848	.625	77	.974	.225	4.331
33	.545	.839	.649	78	.978	.208	4.705
34	.559	.829	.675	79	.982	.191	5.145
35	.574	.819	.700	80	.985	.174	5.671
36	.588	.809	.727	81	.988	.156	6.314
37	.602	.799	.754	82	.990	.139	7.115
38	.616	.788	.781	83	.993	.122	8.144
39	.629	.777	.810	84	.995	.105	9.514
40	.643	.766	.839	85	.996	.087	11.43
41	.656	.755	.869	86	.998	.070	14.30
42	.669	.743	.900	87	.999	.052	19.08
43	.682	.731	.933	88	.999	.035	28.64
44	.695	.719	.966	89	1.000	.017	57.29
45	.707	.707	1.000	90	1.000	.000	

Solutions to Appendix

Problem Set X-1

1. a. $\frac{3}{5}$, $\frac{4}{5}$, $\frac{3}{4}$.

   b. $\frac{5}{13}$, $\frac{12}{13}$, $\frac{5}{12}$.

   c. $\frac{3}{5}$, $\frac{4}{5}$, $\frac{4}{3}$.

   d. $\frac{x}{y}$, $\frac{z}{y}$, $\frac{x}{z}$, $\frac{z}{x}$.

2. a. $\frac{2}{\sqrt{5}}$, b. $\frac{12}{5}$.

3. $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$.

4. $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, 1.

5. a. .34, .94, .36.

   b. .80, .60, 1.33.

Problem Set X-2

2. 43°, 23°, 17°, 24°, 63°, 71°.

3. $\sin 17^\circ = \dfrac{x}{\frac{1}{2}\cdot 5280}$ $x = .292 \cdot 2640 = 771$ feet.

4. $\tan x = \frac{6}{4} = 1.5$. $x = 56^\circ$.

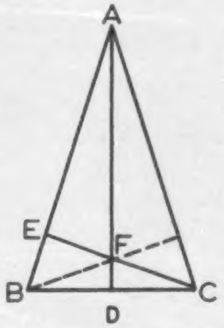
5. $m\angle A = 30$, $m\angle B = m\angle C = 75^\circ$.

   a. $\frac{AD}{CD} = \tan C$. $AD = 3.732 \cdot 3 = 11.196$.

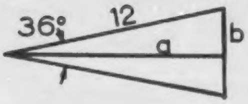
   b. $\frac{CE}{CB} = \sin B$. $CE = .966 \cdot 6 = 5.796$.

   c. $m\angle ECB = 90^\circ - m\angle B = 15^\circ$.

   d. $\frac{DF}{CD} = \tan 15^\circ$. $DF = .268 \cdot 3 = .804$.



6. $\sin 18^\circ = \frac{b}{12}$,



$b = 3.71$, $2b = \underline{7.42}$

$\cos 18^\circ = \frac{a}{12}$,

$a = \underline{11.41}$.

$\text{area} = \frac{1}{2} \cdot 10 \cdot 7.42 \cdot 11.41 = 423$.

7. $\tan 42^\circ = \frac{CD}{50}$, $CD = 45.0$.

$\tan 26^\circ = \frac{45}{AC}$, $AC = 92.2$, $AB = \underline{42.2}$.

$\sin 26^\circ = \frac{45}{AD}$, $AD = \underline{103}$.

Problem Set X-3

1. $(\sin A)^2 + (\cos A)^2 = 1$, $\frac{1}{9} + (\cos A)^2 = 1$,

$\cos A = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$.

$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}} = \frac{1}{2\sqrt{2}}$.

2. (c) is impossible.

(d) A here is congruent to B of part (a).

(g) A here is the complement of the A of part (f).

Appendix XI

REGULAR POLYHEDRA

A polyhedron is a solid whose boundary consists of planar regions -- called faces -- which are polygonal regions. The sides and vertices of the polygons are called the edges and vertices of the polyhedron. Prisms and pyramids are examples of special kinds of polyhedrons. A regular polyhedron is a convex polyhedron (see Section 3-3 for definition of convexity) whose faces are bounded by regular polygons all with the same number of sides and such that there are the same number of faces (and edges) at each vertex. We shall determine all the regular polyhedra, using Euler's famous formula connecting the number of vertices, edges, and faces of a convex polyhedron (more generally, one without any holes). An excellent exposition of this formula can be found in Rademacher and Toeplitz, "The Enjoyment of Mathematics." Strictly speaking, we show that there are only five possibilities for the numbers of vertices, edges, and faces, but omit the proof that each of these possibilities is realized in essentially one and only one way by a regular polyhedron.

Suppose we have a regular polyhedron with V vertices, E edges and F faces, and with r faces about each vertex and n sides (and vertices) for each face. If the E edges were all shrunk slightly, so as to pull away from the vertices, we would have E segments, each with two end-points, and so $2E$ end-points altogether. Now there are r of these end-points near each of the V vertices, and hence rV end-points in all. We must therefore have the relation $rV = 2E$, or

$$V = \frac{2E}{r}. \tag{1}$$

Similarly, imagine each face shrunk and count the resulting sides of the polygonal regions. There are 2 sides near each edge, and so $2E$ sides. There are n sides on each face, and so nF sides. Thus $nF = 2E$, or

$$F = \frac{2E}{n}. \tag{2}$$

Now Euler's formula tells us that

$$V - E + F = 2.$$

Substituting for E and F from Equations (1) and (2), we get

$$\frac{2E}{r} - E + \frac{2E}{n} = 2.$$

Dividing by $2E$ gives

$$\frac{1}{r} - \frac{1}{2} + \frac{1}{n} = \frac{1}{E}. \tag{3}$$

Hence

$$\frac{1}{r} - \frac{1}{2} + \frac{1}{n} > 0,$$

or

$$\frac{1}{r} + \frac{1}{n} > \frac{1}{2}.$$

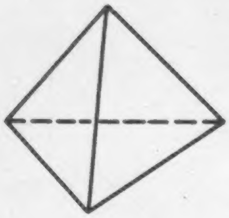
Now $r \geq 3$, so $\frac{1}{r} \leq \frac{1}{3}$, $\frac{1}{n} > \frac{1}{2} - \frac{1}{r} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, so $n < 6$. Thus, $n = 3$, 4, or 5, and the only possibilities for the faces are triangles, squares, or regular pentagons. By the same argument we see that $r = 3$, 4, or 5 are the only possibilities. E can be found from (3), and then V and F from Equations (1) and (2).

For $n = 3$, $r = 3$, we get $V = 4$, $E = 6$, $F = 4$.
For $n = 3$, $r = 4$, we get $V = 6$, $E = 12$, $F = 8$.
For $n = 3$, $r = 5$, we get $V = 12$, $E = 30$, $F = 20$.

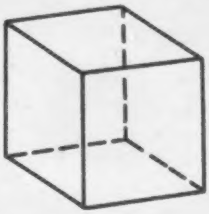
Trying $n = 4$, we see that the only possibility for r is 3, in which case $V = 8$, $E = 12$, $F = 6$. Finally, for $n = 5$, the only possibility is $r = 3$, which yields $V = 20$, $E = 30$, $F = 12$.

These five possibilities are realized in essentially one way for each choice of F, E, and V (more precisely, two regular polyhedra with the same values for F, E, and V are "similar"), although we do not prove this. They are exhibited in the following table:

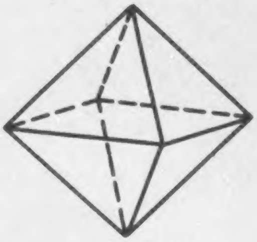
Regular Polyhedron	Boundary of Face	Number of Faces	Number of Edges	Number of Vertices	Number of Faces (or Edges) at a Vertex
Tetrahedron	Triangle	4	6	4	3
Octahedron	Triangle	8	12	6	4
Icosahedron	Triangle	20	30	12	5
Cube (Hexahedron)	Square	6	12	8	3
Dodecahedron	Pentagon	12	30	20	3



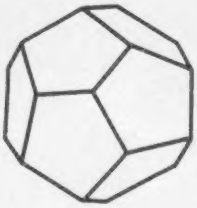
Tetrahedron



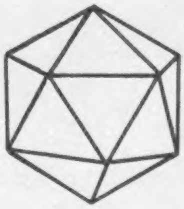
Hexahedron



Octahedron



Dodecahedron



Icosahedron

We observe a curious duality between the octahedron and the cube and between the icosahedron and the dodecahedron, obtained by interchanging F and V, n and r, and leaving E unchanged. The tetrahedron is self-dual. This duality can be established by starting with one of the solids and forming a new one whose vertices are the centers of the faces of the original one, and whose edges are the segments connecting the centers of adjacent faces. These and other relations among the regular polyhedra and related semi-regular polyhedra are discussed in various books; for example, "Mathematical Snapshots," by Steinhaus; "Mathematical Models," by Cundy and Rollett.

The Meaning and Use of Symbols

General.

$=$. $A = B$ can be read as "A equals B", "A is equal to B", "A equal B" (as in "Let $A = B$"), and possibly other ways to fit the structure of the sentence in which the symbol appears. However, we should not use the symbol, $=$, in such forms as "A and B are $=$"; its proper use is *between two expressions*. If two expressions are connected by "$=$" it is to be understood that these two expressions stand for the same mathematical entity, in our case either a real number or a point set.

$\neq$. "Not equal to". $A \neq B$ means that A and B do *not* represent the same entity. The same variations and cautions apply to the use of $\neq$ as to the use of $=$.

Algebraic.

$+$, $\cdot$, $-$, $\div$. These familiar algebraic symbols for operating with real numbers need no comment. The basic postulates about them are presented in Appendix II.

$<$, $>$, $\leq$, $\geq$. Like $=$, these can be read in various ways in sentences, and $A < B$ may stand for the underlined part of "If *A is less than B*", "Let *A be less than B*", "*A less than B* implies", etc. Similarly for the other three symbols, read "greater than", "less than or equal to", "greater than or equal to". These inequalities apply only to real numbers. Their properties are mentioned briefly in Section 2-2, and in more detail in Section 7-2.

$\sqrt{A}$, $|A|$. "Square root of A" and "absolute value of A". Discussed in Sections 2-2 and 2-3 and Appendix IV.

Geometric.

Point Sets. A single letter may stand for any suitably described point set. Thus we may speak of a point P, a line m, a half-plane H, a circle C, an angle x, a segment b, etc.

$\overleftrightarrow{AB}$.	The line containing the two points A and B (P. 30).
$\overline{AB}$.	The segment having A and B as end-points (P.45).
$\overrightarrow{AB}$.	The ray with A as its end-point and containing point B (P. 45).
$\angle ABC$	The angle having B as vertex and $\overrightarrow{BA}$ and $\overrightarrow{BC}$ as sides (P. 71).
$\triangle ABC$.	The triangle having A, B and C as vertices (P. 72).
$\angle$ A-BC-D.	The dihedral angle having $\overleftrightarrow{BC}$ as edge and with sides containing A and D (P. 299).

Real Numbers.

AB.	The positive number which is the distance between the two points A and B, and also the length of the segment $\overline{AB}$ (P. 34).
$m\angle ABC$.	The real number between 0 and 180 which is the degree measure of $\angle ABC$ (P. 80).
Area R.	The positive number which is the area of the polygonal region R (P. 320).

Relations.

$\cong$.	Congruence. $A \cong B$ is read "A is congruent to B", but with the same possible variations and restrictions as A = B. In the text A and B may be two (not necessarily different) segments (P. 109), angles (P. 109), or triangles (P. 111).
$\perp$.	Perpendicular. $A \perp B$ is read "A is perpendicular to B", with the same comment as for $\cong$. A and B may be either two lines (P. 86), two planes (P. 301), or a line and a plane (P. 229).
$\parallel$.	Parallel. $A \parallel B$ is read "A is parallel to B", with the same comment as for $\cong$. A and B may be either two lines (P. 241), two planes (P. 291) or a line and a plane (P. 291).

List of Postulates

Postulate 1. (P. 30) Given any two different points, there is exactly one line which contains both of them.

Postulate 2. (P. 34) (The Distance Postulate.) To every pair of different points there corresponds a unique positive number.

Postulate 3. (P. 36) (The Ruler Postulate.) The points of a line can be placed in correspondence with the real numbers in such a way that

(1) To every point of the line there corresponds exactly one real number,

(2) To every real number there corresponds exactly one point of the line, and

(3) The distance between two points is the absolute value of the difference of the corresponding numbers.

Postulate 4. (P. 40) (The Ruler Placement Postulate.) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

Postulate 5. (P. 54) (a) Every plane contains at least three non-collinear points.

(b) Space contains at least four non-coplanar points.

Postulate 6. (P. 56) If two points lie in a plane, then the line containing these points lies in the same place.

Postulate 7. (P. 57) Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane. More briefly, any three points are coplanar, and any three non-collinear points determine a plane.

Postulate 8. (P. 58) If two different planes intersect, then their intersection is a line.

Postulate 9. (P. 64) (The Plane Separation Postulate.) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that

(1) each of the sets is convex and

(2) if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the line.

Postulate 10. (P. 66) (The Space Separation Postulate.) The points of space that do not lie in a given plane form two sets such that

(1) each of the sets is convex and

(2) if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the plane.

Postulate 11. (P. 80) (The Angle Measurement Postulate.) To every angle $\angle BAC$ there corresponds a real number between 0 and 180.

Postulate 12. (P. 81) (The Angle Construction Postulate.) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $m\angle PAB = r$.

Postulate 13. (P. 81) (The Angle Addition Postulate.) If D is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

Postulate 14. (P. 82) (The Supplement Postulate.) If two angles form a linear pair, then they are supplementary.

Postulate 15. (P. 115) (The S.A.S. Postulate.) Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Postulate 16. (P. 252) (The Parallel Postulatc .) Through a given external point there is at most one line parallel to a given line.

Postulate 17. (P. 320) To every polygonal region there corresponds a unique positive number.

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Postulate 18. (P. 320) If two triangles are congruent, then the triangular regions have the same area.

Postulate 19. (P. 320) Suppose that the region R is the union of two regions R_1 and R_2. Suppose that R_1 and R_2 intersect at most in a finite number of segments and points. Then the area of R is the sum of the areas of R_1 and R_2.

Postulate 20. (P. 322) The area of a rectangle is the product of the length of its base and the length of its altitude.

Postulate 21. (P. 546) The volume of a rectangular parallelepiped is the product of the altitude and the area of the base.

Postulate 22. (P. 548) (Cavalieri's Principle.) Given two solids and a plane. If for every plane which intersects the solids and is parallel to the given plane the two intersections have equal areas, then the two solids have the same volume.

List of Theorems and Corollaries

Theorem 2-1. (P. 42) Let A, B, C be three points of a line, with coordinates x, y, z. If $x < y < z$, then B is between A and C.

Theorem 2-2. (P. 43) Of any three different points on the same line, one is between the other two.

Theorem 2-3. (P. 44) Of three different points on the same line, only one is between the other two.

Theorem 2-4. (P. 46) (The Point Plotting Theorem) Let $\overrightarrow{AB}$ be a ray, and let x be a positive number. Then there is exactly one point P of $\overrightarrow{AB}$ such that $AP = x$.

Theorem 2-5. (P. 47) Every segment has exactly one mid-point.

Theorem 3-1. (P. 55) Two different lines intersect in at most one point.

Theorem 3-2. (P. 56) If a line intersects a plane not containing it, then the intersection is a single point.

Theorem 3-3. (P. 57) Given a line and a point not on the line, there is exactly one plane containing both of them.

Theorem 3-4. (P. 58) Given two intersecting lines, there is exactly one plane containing them.

Theorem 4-1. (P. 87) If two angles are complementary, then both of them are acute.

Theorem 4-2. (P. 87) Every angle is congruent to itself.

Theorem 4-3. (P. 87) Any two right angles are congruent.

Theorem 4-4. (P. 87) If two angles are both congruent and supplementary, then each of them is a right angle.

Theorem 4-5. (P. 87) Supplements of congruent angles are congruent.

Theorem 4-6. (P. 88) Complements of congruent angles are congruent.

Theorem 4-7. (P. 88) Vertical angles are congruent.

Theorem 4-8. (P. 89) If two intersecting lines form one right angle, then they form four right angles.

Theorem 5-1. (P. 109) Every segment is congruent to itself.

Theorem 5-2. (P. 127) If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

Corollary 5-2-1. (P. 128) Every equilateral triangle is equiangular.

Theorem 5-3. (P. 129) Every angle has exactly one bisector.

Theorem 5-4. (P. 132) (The A.S.A. Theorem.) Given a correspondence between two triangles (or between a triangle and itself). If two angles and the included side of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Theorem 5-5. (P. 133) If two angles of a triangle are congruent, the sides opposite these angles are congruent.

Corollary 5-5-1. (P. 133) An equiangular triangle is equilateral.

Theorem 5-6. (P. 137) (The S.S.S. Theorem.) Given a correspondence between two triangles (or between a triangle and itself). If all three pairs of corresponding sides are congruent, then the correspondence is a congruence.

Theorem 6-1. (P. 167) In a given plane, through a given point of a given line of the plane, there passes one and only one line perpendicular to the given line.

Theorem 6-2. (P. 169) The perpendicular bisector of a segment, in a plane, is the set of all points of the plane that are equidistant from the end-points of the segment.

Theorem 6-3. (P. 171) Through a given external point there is at most one line perpendicular to a given line.

Corollary 6-3-1. (P. 172) At most one angle of a triangle can be a right angle.

Theorem 6-4. (P. 172) Through a given external point there is at least one line perpendicular to a given line.

Theorem 6-5. (P. 183) If M is between A and C on a line L, then M and A are on the same side of any other line that contains C.

Theorem 6-6. (P. 183) If M is between A and C, and B is any point not on line $\overleftrightarrow{AC}$, then M is in the interior of $\angle ABC$.

Theorem 7-1. (P. 193) (The Exterior Angle Theorem.) An exterior angle of a triangle is larger than either remote interior angle.

Corollary 7-1-1. (P. 196) If a triangle has a right angle, then the other two angles are acute.

Theorem 7-2. (P. 197) (The S.A.A. Theorem.) Given a correspondence between two triangles. If two angles and a side opposite one of them in one triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Theorem 7-3. (P. 198) (The Hypotenuse - Leg Theorem.) Given a correspondence between two right triangles. If the hypotenuse and one leg of one triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Theorem 7-4. (P. 200) If two sides of a triangle are not congruent, then the angles opposite these two sides are not congruent, and the larger angle is opposite the longer side.

Theorem 7-5. (P. 201) If two angles of a triangle are not congruent, then the sides opposite them are not congruent, and the longer side is opposite the larger angle.

Theorem 7-6. (P. 206) The shortest segment joining a point to a line is the perpendicular segment.

Theorem 7-7. (P. 206) (The Triangle Inequality.) The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Theorem 7-8. (P. 210) If two sides of one triangle are congruent respectively to two sides of a second triangle, and the included angle of the first triangle is larger than the included angle of the second, then the opposite side of the first triangle is longer than the opposite side of the second.

Theorem 7-9. (P. 211) If two sides of one triangle are congruent respectively to two sides of a second triangle, and the third side of the first triangle is longer than the third side of the second, then the included angle of the first triangle is larger than the included angle of the second.

Theorem 8-1. (P. 222) If each of two points of a line is equidistant from two given points, then every point of the line is equidistant from the given points.

Theorem 8-2. (P. 225) If each of three non-collinear points of a plane is equidistant from two points, then every point of the plane is equidistant from these two points.

Theorem 8-3. (P. 226) If a line is perpendicular to each of two intersecting lines at their point of intersection, then it is perpendicular to the plane of these lines.

Theorem 8-4. (P. 230) Through a given point on a given line there passes a plane perpendicular to the line.

Theorem 8-5. (P. 231) If a line and a plane are perpendicular, then the plane contains every line perpendicular to the given line at its point of intersection with the given plane.

Theorem 8-6. (P. 232) Through a given point on a given line there is at most one plane perpendicular to the line.

Theorem 8-7. (P. 232) The perpendicular bisecting plane of a segment is the set of all points equidistant from the end-points of the segment.

Theorem 8-8. (P. 234) Two lines perpendicular to the same plane are coplanar.

Theorem 8-9. (P. 235) Through a given point there passes one and only one plane perpendicular to a given line.

Theorem 8-10. (P. 235) Through a given point there passes one and only one line perpendicular to a given plane.

Theorem 8-11. (P. 235) The shortest segment to a plane from an external point is the perpendicular segment.

Theorem 9-1. (P. 242) Two parallel lines lie in exactly one plane.

Theorem 9-2. (P. 242) Two lines in a plane are parallel if they are both perpendicular to the same line.

Theorem 9-3. (P. 244) Let L be a line, and let P be a point not on L. Then there is at least one line through P, parallel to L.

Theorem 9-4. (P. 246) If two lines are cut by a transversal, and if one pair of alternate interior angles are congruent, then the other pair of alternate interior angles are also congruent.

Theorem 9-5. (P. 246) If two lines are cut by a transverasl, and if a pair of alternate interior angles are congruent, then the lines are parallel.

Theorem 9-6. (P. 252) If two lines are cut by a transversal, and if one pair of corresponding angles are congruent, then the other three pairs of corresponding angles have the same property.

Theorem 9-7. (P. 252) If two lines are cut by a transversal, and if a pair of corresponding angles are congruent, then the lines are parallel.

Theorem 9-8. (P. 253) If two parallel lines are cut by a transversal, then alternate interior angles are congruent.

Theorem 9-9. (P. 254) If two parallel lines are cut by a transversal, each pair of corresponding angles are congruent.

Theorem 9-10. (P. 254) If two parallel lines are cut by a transversal, interior angles on the same side of the transversal are supplementary.

Theorem 9-11. (P. 255) In a plane, two lines parallel to the same line are parallel to each other.

Theorem 9-12. (P. 255) In a plane, if a line is perpendicular to one of two parallel lines it is perpendicular to the other.

Theorem 9-13. (P. 258) The sum of the measures of the angles of a triangle is 180.

Corollary 9-13-1. (P. 259) Given a correspondence between two triangles. If two pairs of corresponding angles are congruent, then the third pair of corresponding angles are also congruent.

Corollary 9-13-2. (P. 260) The acute angles of a right triangle are complementary.

Corollary 9-13-3. (P. 260) For any triangle, the measure of an exterior angle is the sum of the measures of the two remote interior angles.

Theorem 9-14. (P. 265) Either diagonal divides a parallelogram into two congruent triangles.

Theorem 9-15. (P. 265) In a parallelogram, any two opposite sides are congruent.

Corollary 9-15-1. (P. 266) If $L_1 || L_2$ and if P and Q are any two points on L_1, then the distances of P and Q from L_2 are equal.

Theorem 9-16. (P. 266) In a parallelogram, any two opposite angles are congruent.

Theorem 9-17. (P. 266) In a parallelogram, any two consecutive angles are supplementary.

Theorem 9-18. (P. 266) The diagonals of a parallelogram bisect each other.

Theorem 9-19. (P. 266) Given a quadrilateral in which both pairs of opposite sides are congruent. Then the quadrilateral is a parallelogram.

Theorem 9-20. (P. 266) If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

Theorem 9-21. (P. 266) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Theorem 9-22. (P. 267) The segment between the mid-points of two sides of a triangle is a parallel to the third side and half as long as the third side.

Theorem 9-23. (P. 268) If a parallelogram has one right angle, then it has four right angles, and the parallelogram is a rectangle.

Theorem 9-24. (P. 268) In a rhombus, the diagonals are perpendicular to one another.

Theorem 9-25. (P. 268) If the diagonals of a quadrilateral bisect each other and are perpendicular, then the quadrilateral is a rhombus.

Theorem 9-26. (P. 276) If three parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.

Corollary 9-26-1. (P. 277) If three or more parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.

Theorem 9-27. (P. 279) The medians of a triangle are concurrent in a point two-thirds the way from any vertex to the mid-point of the opposite side.

Theorem 10-1. (P. 292) If a plane intersects two parallel planes, then it intersects them in two parallel lines.

Theorem 10-2. (P. 292) If a line is perpendicular to one of two parallel planes it is perpendicular to the other.

Theorem 10-3. (P. 293) Two planes perpendicular to the same line are parallel.

Corollary 10-3-1. (P. 294) If two planes are each parallel to a third plane, they are parallel to each other.

Theorem 10-4. (P. 294) Two lines perpendicular to the same plane are parallel.

Corollary 10-4-1. (P. 294) A plane perpendicular to one of two parallel lines is perpendicular to the other.

Corollary 10-4-2. (P. 294) If two lines are each parallel to a third they are parallel to each other.

Theorem 10-5. (P. 295) Two parallel planes are everywhere equidistant.

Theorem 10-6. (P. 301) Any two plane angles of a given dihedral angle are congruent.

Corollary 10-6-1. (P. 302) If a line is perpendicular to a plane, then any plane containing this line is perpendicular to the given plane.

Corollary 10-6-2. (P. 302) If two planes are perpendicular, then any line in one of them perpendicular to their line of intersection is perpendicular to the other plane.

Theorem 10-7. (P. 307) The projection of a line into a plane is a line, unless the line and the plane are perpendicular.

Theorem 11-1. (P. 328) The area of a right triangle is half the product of its legs.

Theorem 11-2. (P. 328) The area of a triangle is half the product of any base and the altitude to that base.

Theorem 11-3. (P. 330) The area of a parallelogram is the product of any base and the corresponding altitude.

Theorem 11-4. (P. 331) The area of a trapezoid is half the product of its altitude and the sum of its bases.

Theorem 11-5. (P. 332) If two triangles have the same altitude, then the ratio of their areas is equal to the ratio of their bases.

Theorem 11-6. (P. 332) If two triangles have equal altitudes and equal bases, then they have equal areas.

Theorem 11-7. (P. 339) (The Pythagorean Theorem.) In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

Theorem 11-8. (P. 340) If the square of one side of a triangle is equal to the sum of the squares of the other two, then the triangle is a right triangle, with a right angle opposite the first side.

Theorem 11-9. (P. 346) (The 30 - 60 Triangle Theorem.) The hypotenuse of a right triangle is twice as long as the shorter leg if and only if the acute angles are 30° and 60°.

Theorem 11-10. (P. 346) (The Isosceles Right Triangle Theorem.) A right triangle is isosceles if and only if the hypotenuse is $\sqrt{2}$ times as long as a leg.

Theorem 12-1. (P. 368) (The Basic Proportionality Theorem.) If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off segments which are proportional to these sides.

Theorem 12-2. (P. 369) If a line intersects two sides of a triangle, and cuts off segments proportional to these two sides, then it is parallel to the third side.

Theorem 12-3. (P. 374) (The A.A.A. Similarity Theorem.) Given a correspondence between two triangles. If corresponding angles are congruent, then the correspondence is a similarity.

Corollary 12-3-1. (P. 376) (The A.A. Corollary.) Given a correspondence between two triangles. If two pairs of corresponding angles are congruent, then the correspondence is a similarity.

Corollary 12-3-2. (P. 376) If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off a triangle similar to the given triangle.

Theorem 12-4. (P. 376) (The S.A.S. Similarity Theorem.) Given a correspondence between two triangles. If two corresponding angles are congruent, and the including sides are proportional, then the correspondence is a similarity.

Theorem 12-5. (P. 378) (The S.S.S. Similarity Theorem.) Given a correspondence between two triangles. If corresponding sides are proportional, then the correspondence is a similarity.

Theorem 12-6. (P. 391) In any right triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar both to each other and to the original triangle.

Corollary 12-6-1. (P. 392) Given a right triangle and the altitude from the right angle to the hypotenuse:

(1) The altitude is the geometric mean of the segments into which it separates the hypotenuse.

(2) Either leg is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

Theorem 12-7. (P. 395) The ratio of the areas of two similar triangles is the square of the ratio of any two corresponding sides.

Theorem 13-1. (P. 410) The intersection of a sphere with a plane through its center is a circle with the same center and radius.

Theorem 13-2. (P. 414) Given a line and a circle in the same plane. Let P be the center of the circle, and let F be the foot of the perpendicular from P to the line. Then either

(1) Every point of the line is outside the circle, or

(2) F is on the circle, and the line is tangent to the circle at F, or

(3) F is inside the circle, and the line intersects the circle in exactly two points, which are equidistant from F.

Corollary 13-2-1. (P. 416) Every line tangent to C is perpendicular to the radius drawn to the point of contact.

Corollary 13-2-2. (P. 416) Any line in E, perpendicular to a radius at its outer end, is tangent to the circle.

Corollary 13-2-3. (P. 416) Any perpendicular from the center of C to a chord bisects the chord.

Corollary 13-2-4. (P. 416) The segment joining the center of C to the mid-point of a chord is perpendicular to the chord.

Corollary 13-2-5. (P. 416) In the plane of a circle, the perpendicular bisector of a chord passes through the center of the circle.

Corollary 13-2-6 (P. 417) If a line in the plane of a circle intersects the interior of the circle, then it intersects the circle in exactly two points.

Theorem 13-3. (P. 417) In the same circle or in congruent circles, chords equidistant from the center are congruent.

Theorem 13-4. (P. 417) In the same circle or in congruent circles, any two congruent chords are equidistant from the center.

Theorem 13-5. (P. 424) Given a plane E and a sphere S with center P. Let F be the foot of the perpendicular segment from P to E. Then either

(1) Every point of E is outside S, or

(2) F is on S, and E is tangent to S at F, or

(3) F is inside S, and E intersects S in a circle with center F.

Corollary 13-5-1. (P. 426) A plane tangent to S is perpendicular to the radius drawn to the point of contact.

Corollary 13-5-2. (P. 426) A plane perpendicular to a radius at its outer end is tangent to S.

Corollary 13-5-3. (P. 426) A perpendicular from P to a chord of S bisects the chord.

Corollary 13-5-4. (P. 426) The segment joining the center of S to the midpoint of a chord is perpendicular to the chord.

Theorem 13-6. (P. 431) If $\widehat{AB}$ and $\widehat{BC}$ are arcs of the same circle having only the point B in common, and if their union is an arc $\widehat{AC}$, then $m\widehat{AB} + m\widehat{BC} = m\widehat{AC}$.

Theorem 13-7. (P. 434) The measure of an inscribed angle is half the measure of its intercepted arc.

Corollary 13-7-1. (P. 437) An angle inscribed in a semicircle is a right angle.

Corollary 13-7-2. (P. 437) Angles inscribed in the same arc are congruent.

Theorem 13-8. (P. 441) In the same circle or in congruent circles, if two chords are congruent, then so also are the corresponding minor arcs.

Theorem 13-9. (P. 441) In the same circle or in congruent circles, if two arcs are congruent, then so are the corresponding chords.

Theorem 13-10. (P. 442) Given an angle with vertex on the circle formed by a secant ray and a tangent ray. The measure of the angle is half the measure of the intercepted arc.

Theorem 13-11. (P. 448) The two tangent segments to a circle from an external point are congruent, and form congruent angles with the line joining the external point to the center of the circle.

Theorem 13-12. (P. 449) Given a circle C and an external point Q, let L_1 be a secant line through Q, intersecting C in points R and S; and let L_2 be another secant line through Q, intersecting C in points T and U. Then $QR \cdot QS = QU \cdot QT$.

Theorem 13-13. (P. 450) Given a tangent segment $\overline{QT}$ to a circle, and a secant line through Q, intersecting the circle in points R and S. Then $QR \cdot QS = QT^2$.

Theorem 13-14. (P. 451) If two chords intersect within a circle, the product of the lengths of the segments of one equals the product of the lengths of the segments of the other.

Theorem 14-1. (P. 467) The bisector of an angle, minus its end-point, is the set of points in the interior of the angle equidistant from the sides of the angle.

Theorem 14-2. (P. 469) The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the three vertices of the triangle.

Corollary 14-2-1. (P. 470) There is one and only one circle through three non-collinear points.

Corollary 14-2-2. (P. 470) Two distinct circles can intersect in at most two points.

Theorem 14-3. (P. 470) The three altitudes of a triangle are concurrent.

Theorem 14-4. (P. 471) The angle bisectors of a triangle are concurrent in a point equidistant from the three sides.

Theorem 14-5. (P. 476) (The Two Circle Theorem.) If two circles have radii a and b, and if c is the distance between their centers, then the circles intersect in two points, one on each side of the line of centers, provided each one of a, b, c is less than the sum of the other two.

Construction 14-6. (P. 477) To copy a given triangle.

Construction 14-7. (P. 479) To copy a given angle.

Construction 14-8. (P. 481) To construct the perpendicular bisector of a given segment.

Corollary 14-8-1. (P. 481) To bisect a given segment.

Construction 14-9. (P. 482) To construct a perpendicular to a given line through a given point.

Construction 14-10. (P. 484) To construct a parallel to a given line, through a given external point.

Construction 14-11. (P. 484) To divide a segment into a given number of congruent segments.

Construction 14-12. (P. 491) To circumscribe a circle about a given triangle.

Construction 14-13. (P. 491) To bisect a given angle.

Construction 14-14. (P. 492) To inscribe a circle in a given triangle.

Theorem 15-1. (P. 517) The ratio $\frac{C}{2r}$, of the circumference to the diameter, is the same for all circles.

Theorem 15-2. (P. 522) The area of a circle of radius r is πr^2.

Theorem 15-3. (P. 526) If two arcs have equal radii, their lengths are proportional to their measures.

Theorem 15-4. (P. 526) An arc of measure q and radius r has length $\frac{\pi}{180}qr$.

Theorem 15-5. (P. 527) The area of a sector is half the product of its radius by the length of its arc.

Theorem 15-6. (P. 527) The area of a sector of radius r and arc measure q is $\frac{\pi}{360}qr^2$.

Theorem 16-1. (P. 535) All cross-sections of a triangular prism are congruent to the base.

Corollary 16-1-1. (P. 536) The upper and lower bases of a triangular prism are congruent.

Theorem 16-2. (P. 536) (Prism Cross-Section Theorem.) All cross-sections of a prism have the same area.

Corollary 16-2-1. (P. 537) The two bases of a prism have equal areas.

Theorem 16-3. (P. 537) The lateral faces of a prism are parallelogram regions, and the lateral faces of a right prism are rectangular regions.

Theorem 16-4. (P. 540) A cross-section of a triangular pyramid, by a plane between the vertex and the base, is a triangular region similar to the base. If the distance from the vertex to the cross-section plane is k and the altitude is h, then the ratio of the area of the cross-section to the area of the base is $(\frac{k}{h})^2$.

Theorem 16-5. (P. 542) In any pyramid, the ratio of the area of a cross-section and the area of the base is $(\frac{k}{h})^2$, where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross-section.

Theorem 16-6. (P. 543) (The Pyramid Cross-Section Theorem.) Given two pyramids with the same altitude. If the bases have the same area, then cross-sections equidistant from the bases also have the same area.

Theorem 16-7. (P. 548) The volume of any prism is the product of the altitude and the area of the base.

Theorem 16-8. (P. 549) If two pyramids have the same altitude and the same base area, then they have the same volume.

Theorem 16-9. (P. 550) The volume of a triangular pyramid is one-third the product of its altitude and its base area.

Theorem 16-10. (P. 551) The volume of a pyramid is one-third the product of its altitude and its base area.

Theorem 16-11. (P. 555) A cross-section of a circular cylinder is a circular region congruent to the base.

Theorem 16-12. (P. 555) The area of a cross-section of a circular cylinder is equal to the area of the base.

Theorem 16-13. (P. 555) A cross-section of a cone of altitude h, made by a plane at a distance k from the vertex, is a circular region whose area has a ratio to the area of the base of $(\frac{k}{h})^2$.

Theorem 16-14. (P. 557) The volume of a circular cylinder is the product of the altitude and the area of the base.

Theorem 16-15. (P. 557) The volume of a circular cone is one-third the product of the altitude and the area of the base.

Theorem 16-16. (P. 559) The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Theorem 16-17. (P. 562) The surface area of a sphere of radius r is $S = 4\pi r^2$.

Theorem 17-1. (P. 579) On a non-vertical line, all segments have the same slope.

Theorem 17-2. (P. 584) Two non-vertical lines are parallel if and only if they have the same slope.

Theorem 17-3. (P. 586) Two non-vertical lines are perpendicular if and only if their slopes are the negative reciprocals of each other.

Theorem 17-4. (P. 589) (The Distance Formula.) The distance between the points (x_1,y_1) and (x_2,y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Theorem 17-5. (P. 593) (The Mid-Point Formula.) Let $P_1 = (x_1,y_1)$ and let $P_2 = (x_2,y_2)$. Then the mid-point of P_1P_2 is the point $P = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

Theorem 17-6. (P. 605) Let L be a non-vertical line with slope m, and let P be a point of L, with coordinates (x_1,y_1). For every point $Q = (x,y)$ of L, the equation $y - y_1 = m(x - x_1)$ is satisfied.

Theorem 17-7. (P. 606) The graph of the equation $y - y_1 = m(x - x_1)$ is the line that passes through the point (x_1,y_1) and has slope m.

Theorem 17-8. (P. 611) The graph of the equation $y = mx + b$ is the line with slope m and y-intercept b.

Theorem 17-9. (P. 613) Every line in the plane is the graph of a linear equation in x and y.

Theorem 17-10. (P. 613) The graph of a linear equation in x and y is always a line.

Theorem 17-11. (P. 623) The graph of the equation $(x - a)^2 + (y - b)^2 = r^2$ is the circle with center at (a,b) and radius r.

Theorem 17-12. (P. 624) Every circle is the graph of an equation of the form $x^2 + y^2 + Ax + By + C = 0$.

Theorem 17-13. (P. 625) Given the equation $x^2 + y^2 + Ax + By + C = 0$. The graph of this equation is (1) a circle, (2) a point or (3) the empty set.

Index of Definitions

For precisely defined geometric terms the reference is to the formal definition. For other terms the reference is to an informal definition or to the most prominent discussion.

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